§6.3 Normal Principal curvatures

Recall: 1FF is $E\,du^2+2F\,du\,dv+G\,dv^2$, $E=\vec{\sigma}_u\cdot\vec{\sigma}_u$, $F=\vec{\sigma}_u\cdot\vec{\sigma}_v$, $G=\vec{\sigma}_v\cdot\vec{\sigma}_v$
2FF is $L\,du^2+2M\,du\,dv+N\,dv^2$, $L=\vec{\sigma}_{uu}\cdot\vec{N}$, $M=\vec{\sigma}_{uv}\cdot\vec{N}$, $N=\vec{\sigma}_{vv}\cdot\vec{N}$

These encapsulate the behavior/shape of the surface near a point.

Recall: Given unit speed curve $\gamma$ on patch $\vec{\sigma}$, $\ddot{\gamma}=\kappa_n\vec{N}+\kappa_g\vec{N}\times\dot{\gamma}$
defining $\kappa_n$ = normal curvature, $\kappa_g$ = geodesic curvature... also can write $\kappa_n=\ddot{\gamma}\cdot\vec{N}$ and $\kappa_g=\ddot{\gamma}\cdot(\vec{N}\times\dot{\gamma})$
Basically, $\kappa_n$ is related to positioning of surface in space, $\kappa_g$ to the windiness of the curve on the surface patch.

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Prop: Given $\vec{\gamma}(t)=\vec{\sigma}(u(t),v(t))$ unitspeed curve on patch $\vec{\sigma}$, $\kappa_n=L\dot{u}^2+2M\dot{u}\dot{v}+N\dot{v}^2$.
Pf: $\kappa_n=\vec{N}\cdot\ddot{\gamma}=\vec{N}\cdot\frac{d}{dt}(\dot{\gamma})=\vec{N}\cdot(\vec{\sigma}_u\ddot{u}+\vec{\sigma}_v\ddot{v}+(\vec{\sigma}_{uu}\dot{u}+\vec{\sigma}_{uv}\dot{v})\dot{u}+(\vec{\sigma}_{uv}\dot{u}+\vec{\sigma}_{vv}\dot{v})\dot{v})=L\dot{u}^2+2M\dot{u}\dot{v}+N\dot{v}^2$. ∎

Prop (Meusnier's Thm): Given a pt on a surface and a tgt. line, consider the plane $\Pi_\theta$ making an angle $\theta$ w/ the ~~normal~~ tangent plane, passing through P and the tangent line.
If $\kappa_\theta$ is the curvature of the curve $\Pi_\theta\cap S$, then $\kappa_\theta\sin\theta$ is independent of $\theta$.
(?) Pf: $\ddot{\gamma}_\theta$ is parallel to $\Pi_\theta$, so $\kappa_n=\kappa_\theta\sin\theta$.

Let $\mathcal{F}_I=\begin{pmatrix}E&F\\F&G\end{pmatrix}$, $\mathcal{F}_{II}=\begin{pmatrix}L&M\\M&N\end{pmatrix}$. Note that if $T=\begin{pmatrix}\dot{u}\\\dot{v}\end{pmatrix}$ then $\kappa_n=T^t\mathcal{F}_{I}T$.
In general if $\vec{t}_1,\vec{t}_2$ are 2 tangent vectors (in terms of $\vec{\sigma}_u,\vec{\sigma}_v$), then $\vec{t}_1\cdot\vec{t}_2=T_1^t\mathcal{F}_I T_2$.

Def: The principal curvatures of a sfc. patch are the roots of the equation $\det(\mathcal{F}_{II}-\kappa\mathcal{F}_I)=0$,
i.e. $\begin{vmatrix}L-\kappa E & M-\kappa F\\ M-\kappa F & N-\kappa G\end{vmatrix}=0$.
$\Leftrightarrow$ eigenvalues of $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ (its eigenvectors are the principal vectors)

Prop: Let $\kappa_1,\kappa_2$ be the principal curvatures. Then (i) $\kappa_1,\kappa_2\in\mathbb{R}$; (ii) if $\kappa_1=\kappa_2=\kappa$ (P is an "umbilic" point), then $\mathcal{F}_{II}=\kappa\mathcal{F}_I$ and every tangent vector to $\vec{\sigma}$ at P is a principal vector; (iii) if $\kappa_1\neq\kappa_2$, the two principal vectors are perp.
Pf: properties of evals/evects for matrices! ∎

Ex: Sphere: $E=1, F=0, G=\cos^2\theta$, $L=1, M=0, N=\cos^2\theta$ $\Rightarrow$ principals are roots of $\begin{vmatrix}1-\kappa & 0\\ 0 & \cos^2\theta(1-\kappa)\end{vmatrix}=0 \Rightarrow \kappa_1=\kappa_2=1$.
Ex: Cylinder: $E=1, F=0, G=1$, $L=0, M=0, N=1$ $\Rightarrow \begin{vmatrix}0-\kappa & 0\\ 0 & 1-\kappa\end{vmatrix}=0 \Rightarrow \kappa=0,1$, principal vects $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

(Cor 6.1)
Euler's Thm: $\kappa_n=\kappa_1\cos^2\theta+\kappa_2\sin^2\theta$, where $\kappa_n$ is the normal curvature of a curve $\vec{\gamma}$ with $\theta=\angle(\dot{\gamma},\vec{t}_1)$ and $\vec{t}_1$ is the principal vec. of the principal curvature $\kappa_1$.
Consequently, one of $(\kappa_1,\vec{t}_1)$ or $(\kappa_2,\vec{t}_2)$ maximizes the normal curvature, the other minimizes it.

Prop 6.4: For later, note that $\mathcal{F}_I^{-1}\mathcal{F}_{II}=-\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is called the Weingarten matrix. If $\vec{N}$ is the standard unit normal, then $\vec{N}_u=a\vec{\sigma}_u+b\vec{\sigma}_v$ and $\vec{N}_v=c\vec{\sigma}_u+d\vec{\sigma}_v$.

Later sections will study $\vec{N}$, $\vec{N}_u$, and $\vec{N}_v$, so this will come into play later.

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§6.1 Geometry of Principal Curvatures

By rigid motion, we may transform $(P, \vec{t}_1, \vec{t}_2) \rightsquigarrow (O, \binom{1}{0}, \binom{0}{1})$, i.e. an arbitrary point may be translated to the origin w/ principal curvatures along the coordinate axes.

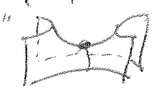
Then $\vec{\sigma}(s,t) \approx (x,y,0) + \frac{1}{2}(s^2\vec{\sigma}_{uu} + 2st\,\vec{\sigma}_{uv} + t^2\vec{\sigma}_{vv}) = (x,y,0) + \frac{1}{2}(s^2 L + 2st\,M + t^2 N)$
and so $\vec{\sigma}(s,t)\cdot\vec{N} \approx z \approx \frac{1}{2}(s\ t)\begin{pmatrix} L & M \\ M & N\end{pmatrix}\binom{s}{t} = \frac{1}{2}(s\ t)\,\mathcal{F}_{II}\binom{s}{t} = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$

Therefore, the surface is $\approx z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$ near a point $P$ w/ principal curvatures $\kappa_1, \kappa_2$.

Classifying the possibilities:

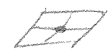
(i) $\kappa_1\kappa_2 > 0$ (both positive or both negative) "elliptic point"

(ii) $\kappa_1\kappa_2 < 0$ (opposite signs) "hyperbolic point"



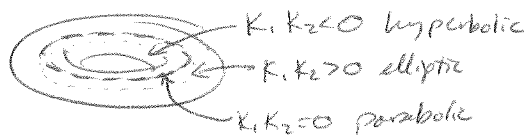
(iii) $\kappa_1\kappa_2 = 0 \longrightarrow$ (iii-a) $\kappa_1$ or $\kappa_2 \neq 0$ "parabolic point"

$\quad\longrightarrow$ (iii-b) $\kappa_1 = \kappa_2 = 0$ "planar point"



Note: Since these depend on only 1st and 2nd derivatives, higher order terms may change the appearance *locally*, e.g. $z = y^4$ is planar.

Ex: Torus:




$\sigma = ((a+b\cos\theta)\cos\varphi,\ (a+b\cos\theta)\sin\varphi,\ b\sin\theta)$

$\kappa_1 = \frac{1}{b},\ \kappa_2 = \frac{\cos\theta}{a+b\cos\theta}$

Thm: If $S$ has every pt. umbilic then the principal curvature is *constant* and $S$ is part of a sphere or a plane.

Pf: $\mathcal{F}_{II} = \kappa\,\mathcal{F}_I$ implies $\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II} = \begin{pmatrix}\kappa & 0 \\ 0 & \kappa\end{pmatrix}$, so $\vec{N}_u = -\kappa\vec{\sigma}_u$, $\vec{N}_v = -\kappa\vec{\sigma}_v$

But $\vec{N}_{uv} = \vec{N}_{vu}$ implies $(\kappa\vec{\sigma}_u)_v = (\kappa\vec{\sigma}_v)_u \Rightarrow \kappa_v\vec{\sigma}_u + \kappa\vec{\sigma}_{uv} = \kappa_u\vec{\sigma}_v + \kappa\vec{\sigma}_{uv} \Rightarrow \kappa_v\vec{\sigma}_u = \kappa_u\vec{\sigma}_v$
$\Rightarrow \kappa_u = \kappa_v = 0$ everywhere. So $\kappa$ is constant!
Now $\kappa = 0$ implies planar since $(\vec{N}\cdot\vec{\sigma})_u = (\vec{N}\cdot\vec{\sigma})_v = 0$ and $\vec{N}\cdot\vec{\sigma} = C$. plane
But $\kappa \neq 0$ implies $\vec{N} = -\kappa\vec{\sigma} + \vec{a}$ and $\|\vec{\sigma} - \frac{1}{\kappa}\vec{a}\|^2 = \|-\frac{1}{\kappa}\vec{N}\|^2 = \frac{1}{\kappa^2} \Rightarrow$ sphere w/ center $\frac{1}{\kappa}\vec{a}$, radius $\frac{1}{\kappa}$. $\square$