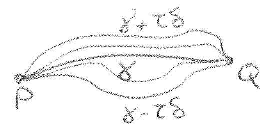
Geodesics—Shortest Paths

§8.4 Variational approach: $\vec\gamma$ is length-minimizing if $\frac{d}{d\tau}\Big(\underbrace{\int_a^b \|\dot\gamma + \tau\dot\delta\|\,dt}_{\mathcal{L}(\tau)}\Big)\Big|_{\tau=0} = 0$, for all δ.



Thm: Unit-speed γ is geodesic iff $\left(\frac{d}{d\tau}\mathcal{L}(\tau)\right)\Big|_{\tau=0} = 0$ for all curves δ.

Pf: Let γ be a geodesic.
We know that $\|\dot\gamma\|=1$ and $\ddot\gamma \perp$ surface for all t, $\tau=0$.
So WTS that the variational equation is equivalent to $\ddot\gamma \perp$ tangent plane for γ unit-speed.

Now $\frac{d}{d\tau}\mathcal{L}(\tau) = \frac{d}{d\tau}\int_a^b f(t,\tau)\,dt = \int_a^b \frac{\partial f}{\partial \tau}\,dt = \int_a^b \frac{\partial}{\partial\tau}\left(\sqrt{g(t,\tau)}\right)dt = \frac{1}{2}\int_a^b \frac{1}{\sqrt{g(t,\tau)}}\frac{\partial g}{\partial \tau}\,dt$

where $g(t,\tau) = E^{(\tau)}\dot u^2 + 2F^{(\tau)}\dot u\dot v + G(\tau)\dot v^2$, dot represents $\frac{d}{dt}$.

Can write $\dfrac{\frac{1}{2}\left(\frac{\partial g}{\partial \tau}\right)}{\sqrt{g(t,\tau)}} = \mathcal{U}\frac{\partial u}{\partial \tau} + \mathcal{V}\frac{\partial v}{\partial \tau}$ where $\mathcal{U}(t,\tau) = \frac{1}{2}g^{-1/2}(E_u\dot u^2 + 2F_u\dot u\dot v + G_u\dot v^2) - \frac{d}{dt}\left(\frac{1}{\sqrt g}(E\dot u + F\dot v)\right)$
$\mathcal{V}(t,\tau) = \frac{1}{2}g^{-1/2}(E_v\dot u^2 + 2F_v\dot u\dot v + G_v\dot v^2) - \frac{d}{dt}\left(\frac{1}{\sqrt g}(F\dot u + G\dot v)\right)$

0 for geodesics via (✱), since $g=1$ for unit-speed curve at $\tau=0$.

$\therefore$ Geodesic $\Rightarrow \frac{d\mathcal{L}}{d\tau}\Big|_{\tau=0} = 0$.

Converse: $\int_a^b \left(\mathcal{U}\frac{\partial u}{\partial \tau} + \mathcal{V}\frac{\partial v}{\partial\tau}\right)dt = 0$ when $\tau=0$ $\forall$ curves $\Rightarrow \mathcal{U}=\mathcal{V}=0$ at $\tau=0$. ∎

Rmks: ① Shortest path $\Rightarrow$ geodesic but not conversely
② Shortest paths always exist on closed sfcs, but not necessarily on open ones.
③ Geodesic Paths on closed sfcs may be extended indefinitely.

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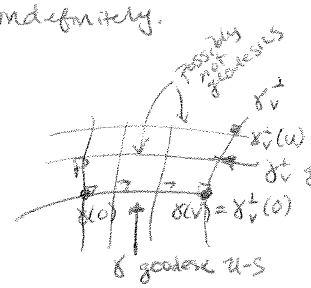
§8.5 Geodesic Coordinates

geodesic paths outward
"geodesic circle" of small radius.



Prop: $\vec\sigma$ is a sfc. patch (for small enough piece of sfc) w/ $\mathcal{F}_I = du^2 + G(u,v)dv^2$,
for some smooth G w/ $G(0,v)=1$, $G_u(0,v)=0$.

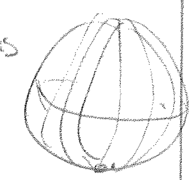
Pf: $\vec\sigma_u(0,v) = \frac{d}{du}\gamma_v^\perp(u)\Big|_{u=0}$, $\vec\sigma_v(0,v) = \frac{d}{dv}\gamma_v^\perp(0) = \frac{d}{dv}\gamma(v)$

perp. by construction.

$\therefore$ Jacobian $\begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v\end{pmatrix}$ has rank 2 at $(0,0)$, hence invertible wrt some coords near the origin,
ie. $\vec\sigma_u, \vec\sigma_v$ are linearly independent on some neighborhood. ✓

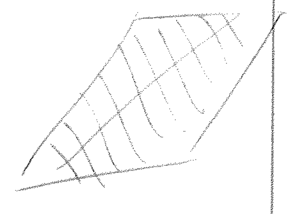
Secondly: $E = \|\vec\sigma_u\|^2 = \left\|\frac{d}{du}\gamma_v^\perp(u)\right\|^2 = 1$ since $\gamma_v^\perp$ is unit-speed.
$F=0$ at $u=0$ since $\vec\sigma_u \perp \vec\sigma_v$ there. But $F_u = 0$ by (✱2) since $\dot v=0$, $\dot u=1$, $t=u$ along the $\gamma_v^\perp$ geodesic.

$G(0,v) = \|\sigma_v(0,v)\|^2 = \left\|\frac{d\gamma}{dv}\right\|^2 = 1$ since γ is u-s,
applying (✱1) we get $G_u(0,v) = 0$. ∎

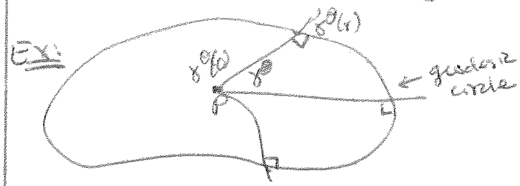


Ex:
P on equator of sphere, γ = equator w/ param. longitude $\Rightarrow$ $\gamma_v^\perp$ are meridians
$\Rightarrow$ geodesic patch $d\theta^2 + \cos^2\theta\, d\varphi^2$.

Existence/Uniqueness $\Rightarrow$ any patch w/ $\perp$ geodesics is a geodesic patch.



Ex:



"Polar geodesic coords"

Has $\mathcal{F}_I = dr^2 + G(r,\theta)\,d\theta^2$.

Ex: Sfc. revol.