Geodesics II 7-Apr-09 ①

Review:

Curve is geodesic: $\ddot{\gamma}$ always $\perp$ to tangent plane (ie parallel to $\vec{N}$)

Facts: ① Geodesics have constant speed
② Geodesics have $\kappa_g = 0$ (zero geod. curvature)
③ Straight lines + normal sections are geodesic.
④ Geodesics are preserved by isometries
⑤ For each pt $P \in \Sigma$ and tangent $\vec{T} \in T_P\Sigma$, there is a geodesic thru P w/ tangent $\vec{T}$.

Geodesic Eqns: $\boxed{\begin{aligned} &\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2) \\ &\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2) \end{aligned}}$ (✱)

$\boxed{\begin{aligned} &\text{Unit-Speed Condition:} \\ &E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = 1 \end{aligned}}$ (▽)

8.3 Sfc. of Revol. $\vec{\sigma}(u,v) = \langle f(u)\cos v,\ f(u)\sin v,\ g(u) \rangle$ $\Rightarrow$ $\mathcal{F}_I = du^2 + f(u)^2 dv^2$ p.125

$\left(\|\tfrac{d}{du}\langle f, g \rangle\| = 1,\ f > 0 \right)$ $\qquad E=1 \quad F=0 \quad G=f(u)^2$

So (✱) becomes $\ddot{u} = \frac{1}{2}(0 + 0 + 2f\frac{df}{du}\dot{v}^2) = f(u)\frac{df}{du}\dot{v}^2$ (A) (▽) becomes $\dot{u}^2 + f(u)^2\dot{v}^2 = 1$. (C)

$\frac{d}{dt}(f(u)^2\dot{v}) = 0$ (B)

So (i) every meridian (v=const.) is geodesic ... b/c $\dot{v} = 0 \Rightarrow$ (B) true, $\dot{u}^2 = 1 \underset{(C)}{\Rightarrow} \ddot{u} = 0 \Rightarrow$ (A) true. ✓

(ii) parallel ($u = u_0$) is geodesic iff $\frac{df}{du} = 0$, ie. u_0 is "stationary pt" of f.

b/c (A) means $f(u_0)\frac{df}{du}\dot{v}^2 = 0$, (C) means $f(u_0)^2\dot{v}^2 = 1$ $\Rightarrow \frac{df}{du} = 0$.

But $\dot{v} = \pm 1/f(u_0)$ = const so (B) is true. ✓

Clairaut's Thm: If ρ is distance from pt to axis of rotation, ψ is the angle between a geodesic γ and the meridians, then $\rho \sin\psi$ is constant along γ.
Conversely, $\rho\sin\psi$ and no overlap w/ a parallel implies geodesic.

Proof: $\rho = f(u)$, and the tangent space is spanned by $\frac{\vec{\sigma}_u}{\|\sigma_u\|} = \vec{\sigma}_u$ and $\frac{\vec{\sigma}_v}{\|\sigma_v\|} = \frac{1}{\rho}\vec{\sigma}_v$, which are perp. unit vectors.

So for γ unit-speed, $\dot{\gamma} = \cos\psi\,\vec{\sigma}_u + \frac{1}{\rho}\sin\psi\,\vec{\sigma}_v$ $\Rightarrow$ $\vec{\sigma}_u \times \dot{\gamma} = \frac{1}{\rho}\sin\psi\,\vec{\sigma}_u \times \vec{\sigma}_v$,
or $\vec{\sigma}_u \times (\dot{u}\vec{\sigma}_u + \dot{v}\vec{\sigma}_v) = \dot{v}\,\vec{\sigma}_u \times \vec{\sigma}_v = \frac{1}{\rho}\sin\psi\,\vec{\sigma}_u \times \vec{\sigma}_v$
$\Rightarrow \dot{v} = \frac{1}{\rho}\sin\psi$

Now $\frac{d}{dt}(\rho^2\dot{v}) \overset{(B)}{=} \frac{d}{dt}(\rho\sin\psi) = 0$ $\Rightarrow \rho\sin\psi$ = constant. ✓

Conversely, if $\rho\sin\psi = \Omega$ then $\dot{v} = \frac{\Omega}{\rho^2}$ (→ automatically implies (B)), $\dot{u}^2 = 1 - \frac{\Omega^2}{\rho^2} \Rightarrow 2\dot{u}\ddot{u} = \frac{2\Omega^2}{\rho^3}\dot{\rho} = \frac{2\Omega^2}{\rho^3}\frac{d\rho}{du}\dot{u}$

so that $\dot{u}(\ddot{u} - \rho\frac{d\rho}{du}\dot{v}^2) = 0$

Since $\dot{u} \neq 0$ in any interval, must have $\ddot{u} = \rho\frac{d\rho}{du}\dot{v}^2$ implying (A). ∎

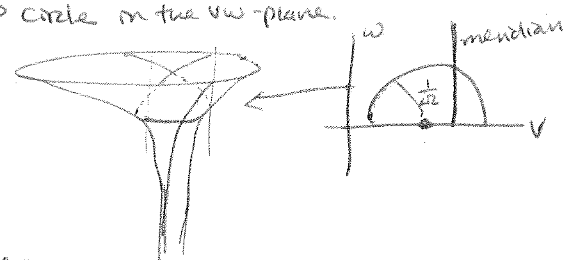
Ex: Pseudosphere $\vec{\sigma}(u,v) = \langle e^{u}\cos v,\ e^{u}\sin v,\ \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})\rangle$, $\mathcal{F}_I = du^2 + e^{2u}dv^2$.
$\rightsquigarrow$ reparam $w = e^{-u}$: $\tilde{\sigma}(v,w) = \langle \frac{1}{w}\cos v,\ \frac{1}{w}\sin v,\ \sqrt{1-\frac{1}{w^2}} - \cosh^{-1} w\rangle$, $\mathcal{F}_I = \frac{1}{w^2}(dv^2 + dw^2)$

Require $w > 1$ for smoothness.

Unit speed geodesic $\Rightarrow$ $\dot{v}^2 + \dot{w}^2 = w^2$
Clairaut's Thm $\Rightarrow$ $\frac{1}{w}\sin\psi = \frac{1}{w^2}\dot{v} = \Omega$ since $\rho = \frac{1}{w}$. So $\dot{v} = \Omega w^2$.

For $\Omega = 0$: get meridian $v = \text{constant}$.
For $\Omega \neq 0$: get $\dot{w} = \pm w\sqrt{1-\Omega^2 w^2}$ so $\frac{dv}{dw} = \frac{\dot{v}}{\dot{w}} = \pm\frac{\Omega w}{\sqrt{1-\Omega^2 w^2}}$ and integrating $v - v_0 = \mp\frac{1}{\Omega}\sqrt{1-\Omega^2 w^2}$.
Thus $(v-v_0)^2 + w^2 = \frac{1}{\Omega^2}$ $\Leftrightarrow$ circle in the vw-plane.

Since $\rho = \frac{1}{w}$, implies geodesic has $\frac{1}{\rho} \leq \frac{1}{\Omega}$ or $\rho \geq \Omega$, so is ~~tangent~~ outside to cylinder about axis of radius Ω.



__In general:__ Clairaut's implies $\rho\sin\psi = \Omega$ ← call it angular momentum.

Assume positive angles, i.e. $\Omega > 0$.
Then $\dot{u}^2 = 1 - \frac{\Omega^2}{\rho^2}$ implies $\rho \geq \Omega$ in general, i.e. contained outside cylinder of radius Ω.

Without equality somewhere, $\dot{u} > 0$ everywhere and the geodesic crosses all parallels.

Ex: Hyperboloid: