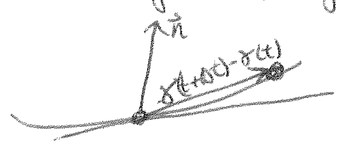
Chapter 6 - Curvature

Second Fundamental Form.

§6.1 2FF

$\boxed{\text{2FF: } L=\vec{\sigma}_{uu}\cdot\vec{N},\ M=\vec{\sigma}_{uv}\cdot\vec{N},\ N=\vec{\sigma}_{vv}\cdot\vec{N},\ \text{recall } \vec{N}=\frac{\vec{\sigma}_u\times\vec{\sigma}_v}{\|\vec{\sigma}_u\times\vec{\sigma}_v\|}. \Rightarrow L\,du^2+2M\,du\,dv+N\,dv^2}$

Curve Derivation: $(\vec{\gamma}(t+\Delta t)-\vec{\gamma}(t))\cdot\vec{n}$ measures tendency of $\vec{\gamma}$ to move away from its tangent line.



Taylor's $\Rightarrow$ this is $(\dot{\vec{\gamma}}\Delta t+\frac{1}{2}\ddot{\vec{\gamma}}(\Delta t)^2+O(\Delta t^3))\cdot\vec{n}$

$= \frac{1}{2}\kappa(\Delta t)^2+O(\Delta t^3)$

So κ measures this tendency.

Sfc Derivation: $(\vec{\sigma}(u+\Delta u,v+\Delta v)-\vec{\sigma}(u,v))\cdot\vec{N}=(\sigma_u\Delta u+\sigma_v\Delta v+\frac{1}{2}\sigma_{uu}\Delta u^2+\sigma_{uv}\Delta u\Delta v+\frac{1}{2}\sigma_{vv}\Delta v^2+O(\Delta u^2,\Delta v^2))\cdot\vec{N}$

$$=\frac{1}{2}\underbrace{\left(L\,\Delta u^2+2M\,\Delta u\Delta v+N\,\Delta v^2\right)}_{\text{measures deviation from tangent plane.}}+O(\Delta u^2,\Delta v^2)\cdot\vec{N}$$

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§6.2 Curves on the Surface

Let $\vec{\gamma}=\vec{\sigma}(u,v)$ be a unit speed curve on patch σ.

$\Rightarrow\|\dot{\vec{\gamma}}\|=1$, $\dot{\vec{\gamma}}$ is in tangent plane $\Rightarrow \dot{\vec{\gamma}}\cdot\vec{N}=0\Rightarrow\{\dot{\vec{\gamma}},N,N\times\dot{\vec{\gamma}}\}$ is an O.N. set.

Thus, $\ddot{\vec{\gamma}}\cdot\dot{\vec{\gamma}}=0$ implies $\ddot{\vec{\gamma}}=\kappa_n\vec{N}+\kappa_g\vec{N}\times\dot{\vec{\gamma}}$ where $\boxed{\begin{array}{l}\kappa_n=\text{normal curvature}\\ \kappa_g=\text{geodesic curvature}\end{array}}$ ← curvature out of tangent plane. ← curvature inside the plane

One computes $\kappa_n=\ddot{\vec{\gamma}}\cdot\vec{N}$, $\kappa_g=\ddot{\vec{\gamma}}\cdot(\vec{N}\times\dot{\vec{\gamma}})$, $\|\ddot{\vec{\gamma}}\|^2=\kappa_n^2+\kappa_g^2$,

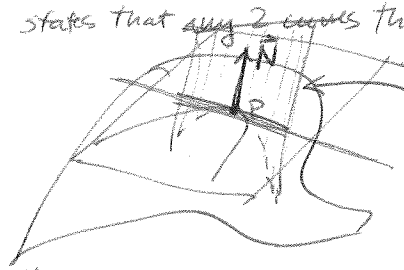
In terms of curve's standard curvature, $\kappa^2=\kappa_n^2+\kappa_g^2$, $\left.\begin{array}{l}\kappa_n=\kappa\,\vec{n}\cdot\vec{N}=\kappa\cos\psi\\ \kappa_g=\pm\kappa\sin\psi\end{array}\right\}$ ψ is angle btwn $\vec{n}$ and $\vec{N}$.

Rmk: κ_n is indep. of parametrization, κ_g may change sign.

A normal section of the sfc. is the intersection of the sfc. w/ a plane $\perp$ to a tangent plane thru $\vec{\gamma}$.

Then $\vec{n}$ and $\vec{N}$ are parallel so $\kappa_n=\pm\kappa$, $\kappa_g=0$.

Meusnier Thm states that ~~any 2 curves~~ thru p w/ same tgt. vector have same normal curvature.



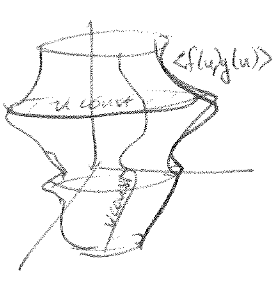
We will eventually write out both κ_n and κ_g in terms of the 2FF.

Ex: Sfc. of Revol. $\vec{\sigma}(u,v)=\langle f(u)\cos v, f(u)\sin v, g(u)\rangle$. w/ $\|\langle\dot{f},\dot{g}\rangle\|=1$ (unit speed), $f>0$

$\vec{\sigma}_u=\langle\dot{f}\cos v,\dot{f}\sin v,\dot{g}\rangle$ $\quad\vec{\sigma}_u\times\vec{\sigma}_v=\langle -f\dot{g}\cos v,-f\dot{g}\sin v,f\dot{f}\rangle$

$\vec{\sigma}_v=\langle -f\sin v,f\cos v,0\rangle$ $\quad\|\sigma_u\times\sigma_v\|=f$.

$E=\|\sigma_u\|^2=\dot{f}^2+\dot{g}^2=1$, $F=0$, $G=\|\sigma_v\|^2=f^2$ $\quad\vec{N}=\langle -\dot{g}\cos v,-\dot{g}\sin v,\dot{f}\rangle$



$L=\dot{f}\ddot{g}-\ddot{f}\dot{g}$, $M=0$, $N=f\dot{g}$

$\Rightarrow 2FF=(\dot{f}\ddot{g}-\ddot{f}\dot{g})du^2+f\dot{g}\,dv^2$.

$\left\{\begin{array}{l}\sigma_{uu}=\langle\ddot{f}\cos v,\ddot{f}\sin v,\ddot{g}\rangle\\ \sigma_{uv}=\langle -\dot{f}\sin v,\dot{f}\cos v,0\rangle\\ \sigma_{vv}=\langle -f\cos v,-f\sin v,0\rangle\end{array}\right.$

E.g. Cylinder $\langle\cos v,\sin v,u\rangle$ $f=1$, $g=u$

$\Rightarrow 2FF=0\,du^2+dv^2=dv^2$ (curvature only in v direction)

Curve on Cylinder: $\vec{\gamma}=\langle\cos at,\sin at,bt\rangle$: magn. $\sqrt{a^2+b^2}$ assume $a^2+b^2=1$.

$\dot{\vec{\gamma}}=\langle -a\sin at,a\cos at,b\rangle$
$\ddot{\vec{\gamma}}=\langle -a^2\cos at,-a^2\sin at,0\rangle$

$\vec{N}=\langle -\cos at,-\sin at,0\rangle$
so $\kappa_n=a^2$, $\kappa_g=0$