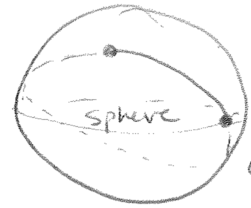
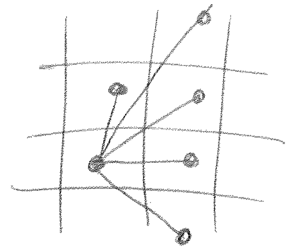
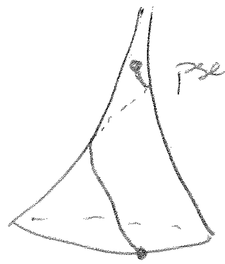
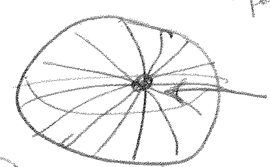
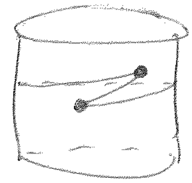
8.1 Geodesics

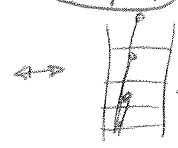


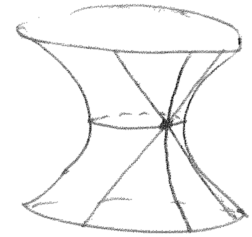












Defn: Curve $\gamma \in \Sigma$ geodesic $\iff \ddot{\gamma}(t) \parallel$ unit normal $\iff \ddot{\gamma}(t)=0$ or $\perp$ surface

Prop 8.1: Geodesics have constant speed.

Pf: $\frac{d}{dt}\|\dot{\gamma}\|^2 = 2\dot{\gamma}\cdot\ddot{\gamma} = 0.$ ■

Prop 8.2: Recall (§6.2, p.127) that $\ddot{\vec{\gamma}} = \underbrace{\kappa_n}_{\text{"normal curvature"}} \vec{N} + \underbrace{\kappa_g}_{\text{"geodesic curvature"}} \vec{N}\times\dot{\vec{\gamma}}$.

Then a curve is a geodesic $\iff \kappa_g = 0$ everywhere.

$$\begin{cases} \kappa_n = \ddot{\gamma}\cdot\vec{N} \\ \kappa_g = \ddot{\gamma}\cdot(\vec{N}\times\dot{\gamma}) \\ \kappa^2 = \kappa_n^2 + \kappa_g^2 \end{cases}$$

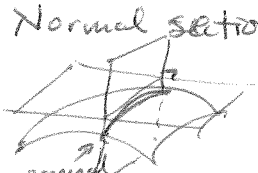
Pf: Clear from the equation, since $\vec{N} \perp \vec{N}\times\ddot{\gamma}$. ■

Prop 8.3: Straight lines are always geodesics.

Pf: $\gamma = \vec{a} + b\vec{t} \Rightarrow \ddot{\gamma} = 0.$ ■

Prop 8.4: Normal sections of a surface are geodesics.

Pf:



The curve must have $\ddot{\gamma} \parallel \vec{N}$ since it must be $\perp$ to the tangent plane. ■

Ex: Great circles are geodesics (since they are normal sections)

Intersections of general cylinders w/ plane ~~perp~~. rulings are geodesic, since normals are $\perp$ rulings.

8.2 Geodesic Equations

Thm: γ on Σ is geodesic $\iff$ for $\gamma(t) = \sigma(u(t), v(t))$ on σ patch,

- $\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$
- $\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$

(*)

Pf: Geodesics have $\ddot{\gamma} \perp$ Tgt Plane $\Rightarrow \ddot{\gamma}\cdot\vec{\sigma}_u = 0$, $\ddot{\gamma}\cdot\vec{\sigma}_v = 0$, or $\left(\frac{d}{dt}\overbrace{(\dot{u}\vec{\sigma}_u + \dot{v}\vec{\sigma}_v)}^{\dot{\gamma}}\right)\cdot\vec{\sigma}_u = 0$, same for v

These are equivalent to the geodesic equations. ■

Note: We are trying to solve (*) for the unknowns $u(t), v(t)$... these are very tough diff eqs!

<u>Ex:</u> Sphere $\sigma = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$ $\mathcal{F}_I = d\theta^2 + \cos^2\theta\, d\varphi^2$, so $E=1, F=0, G=\cos^2\theta$

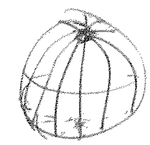
$E_\theta = E_\varphi = 0$
$F_\theta = F_\varphi = 0$
$G_\theta = -2\sin\theta\cos\theta \quad G_\varphi = 0.$

So (*) becomes $\frac{d}{dt}(\dot\theta) = \frac{1}{2}(-2\sin\theta\cos\theta\,\dot\varphi^2)$

$\frac{d}{dt}(\cos^2\theta\,\dot\varphi) = \frac{1}{2}(0)$

So $\cos^2\theta\,\dot\varphi = \Omega$ constant.

If $\Omega = 0$, then $\dot\varphi = 0 \Rightarrow \varphi$ = constant $\Rightarrow$ meridian



Else $\dot\varphi = \frac{\Omega}{\cos^2\theta}$

$\Rightarrow \underbrace{\dot\theta^2 = 1-\dot\varphi^2\cos^2\theta = 1 - \frac{\Omega^2}{\cos^2\theta}}_{\text{unit speed requirement: } \dot\theta^2 + \dot\varphi^2\cos^2\theta = 1}$ so $\left(\frac{d\varphi}{d\theta}\right)^2 = \left(\frac{\dot\varphi^2}{\dot\theta^2}\right) = \frac{\Omega^2/\cos^4\theta}{1-\Omega^2/\cos^2\theta} = \frac{1}{\cos^4\theta\,\Omega^{-2} - \cos^2\theta}$ so $\frac{d\varphi}{d\theta} = \frac{\pm 1}{\cos\theta\sqrt{\cos^2\theta/\Omega^2 - 1}}$

Thus $\varphi = \varphi_0 \pm \int \frac{d\theta}{\cos\theta\sqrt{\cos^2\theta/\Omega^2 - 1}} = \varphi_0 \pm \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\frac{1}{\Omega^2} - 1}}\right)$ or $\tan\theta = \left(\pm\sqrt{1/\Omega^2 - 1}\right)\sin(\varphi - \varphi_0)$

So $z = (\mp\sqrt{1/\Omega^2 - 1}\,\sin\varphi_0)\,x + (\pm\sqrt{1/\Omega^2 - 1}\,\cos\varphi_0)\,y$ is a plane thru the origin, hence a great circle. ∎

<u>Cor:</u> For each $P \in \Sigma$, tgt. vector $\vec{t} \in T_P\Sigma$, $\exists!$ unit-speed geodesic on Σ thru P w/ this tangent vec.

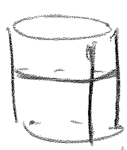
<u>Pf:</u> Since (*) has the form $\ddot{u} = f(u, v, \dot u, \dot v)$, $\ddot{v} = g(u, v, \dot u, \dot v)$, where f, g are smooth, existence and uniqueness of Diff Eqs implies a unique solution (locally), given initial conditions for $u, v, \dot u, \dot v$.
But specifying the pt + tangent vector sets these conditions. ∎

<u>Ex:</u> ∴ straight lines in the plane are the only geodesics.
great circles on sphere are the only geodesics.

~~Ex~~ (Cor 32)
<u>Prop:</u> Isometries between surfaces preserve geodesics.

<u>Pf:</u> Since E, F, G are preserved, so are the geodesic eqns — ie. $\vec\gamma$ satisfies (*) iff $f\circ\vec\gamma$ does also. ∎

<u>Ex:</u> Cylinder



Know straight line/circles are geodesics.

Unwrapping onto the plane implies images of straight lines (ie helices) are also geodesics, and in fact <u>all</u> of them!
The same can be applied to the cone.