Chapter 7 – Gaussian Curvature

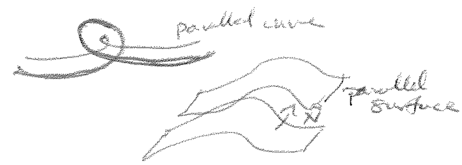
7.4 Suppose $H = \frac{1}{2}(\kappa_1+\kappa_2)$ is constant $\Longleftrightarrow$ minimal surfaces..

$$= \frac{LG-2MF+NE}{2(EG-F^2)}$$

(because variation of sfc. area can be written as an integral over H)

any sfc. minimizing area for a given boundary is a minimal sfc.

Def: Given $\vec{\sigma}$ patch, $\vec{\sigma}^\lambda = \vec{\sigma} + \lambda \vec{N}$ is a parallel surface



Prop: Spse. $|\kappa_1|, |\kappa_2| \le C$ everywhere.

Let λ be constant s.t. $|\lambda| < \frac{1}{C}$ everywhere. Then

(i) $\vec{\sigma}^\lambda$ is a reg'lr sfc. patch

(ii) $\vec{N}^\lambda = \vec{N}$ $\forall (u,v)$ ... same normals

Keeps surface from self intersections.

(iii) principal curvatures are $\frac{\kappa_1}{1-\lambda\kappa_1}$, $\frac{\kappa_2}{1-\lambda\kappa_2}$

(iv) $K^\lambda = K(1-2\lambda H+\lambda^2 K)^{-1}$

$H^\lambda = (H-\lambda K)(1-2\lambda H+\lambda^2 K)^{-1}$

Pf: $\vec{\sigma}_u^\lambda = \vec{\sigma}_u + \lambda \vec{N}_u = (1+\lambda a)\vec{\sigma}_u + \lambda b \vec{\sigma}_v$ ... $\rightarrow I\binom{\sigma_u}{\sigma_v} + \lambda W\binom{N_u}{N_v}$

$\vec{\sigma}_v^\lambda = \vec{\sigma}_v + \lambda \vec{N}_v = \lambda c\, \vec{\sigma}_u + (1+\lambda d)\vec{\sigma}_v$ ... where $W = -\begin{pmatrix} a & c \\ b & d\end{pmatrix}$

$\Rightarrow \vec{\sigma}_u^\lambda \times \vec{\sigma}_v^\lambda \parallel \vec{\sigma}_u \times \vec{\sigma}_v$ so $\vec{N}^\lambda = \vec{N}$

$\cancel{\binom{\sigma_u}{\sigma_v} = W\binom{N_u}{N_v}}$ $\cancel{\binom{\sigma_u^\lambda}{\sigma_v^\lambda} = -W^\lambda\binom{N_u^\lambda}{N_v^\lambda}}$ $\cancel{= I\binom{\sigma_u}{\sigma_v} + \lambda W\binom{N_u}{N_v}}$ $\Rightarrow$ $\cancel{= W\binom{N_u}{N_v} + \lambda W\binom{N_u}{N_v}}$

In particular $\vec{\sigma}_u^\lambda \times \vec{\sigma}_v^\lambda = (1-\lambda\kappa_1)(1-\lambda\kappa_2)\vec{\sigma}_u \times \vec{\sigma}_v$ ... and the constants $|\lambda\kappa_i| < 1$ so this is nonzero $\Rightarrow$ regular

We have $$\begin{pmatrix}\vec{N}_u \\ \vec{N}_v\end{pmatrix} = -W\begin{pmatrix}\vec{\sigma}_u \\ \vec{\sigma}_v\end{pmatrix} = -W^\lambda\begin{pmatrix}\vec{\sigma}_u^\lambda \\ \vec{\sigma}_v^\lambda\end{pmatrix} = -W^\lambda\left[\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix} + \lambda\begin{pmatrix}N_u \\ N_v\end{pmatrix}\right] = -W^\lambda\left[\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix} - \lambda W\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix}\right]$$

$$\Rightarrow -W = -W^\lambda(I-\lambda W) \Rightarrow W^\lambda = W(I-\lambda W)^{-1}$$

So evals. of W are evals of $I-\lambda W$, hence evals of W^λ, and $\kappa \mapsto \frac{\kappa}{1-\lambda\kappa}$. ∎

Cor: If H is constant, $\lambda = \frac{1}{2H}$, then $\vec{\sigma}^\lambda$ has constant Gaussian curvature $4H^2$.

Conversely, if $\vec{\sigma}$ has constant Gaussian curvature K, then for $\lambda = \frac{\pm 1}{\sqrt{K}}$, $\vec{\sigma}^\lambda$ has constant $H = \mp\frac{1}{2}\sqrt{K}$.

$\therefore$ Constant Gaussian cvtr $\longleftrightarrow$ constant mean cvtr. under the transformation of parallel surfaces.

7.5 **Recall:** $\begin{cases} K>0 & \text{elliptic} \\ K=0 & \text{parabolic or planar} \\ K<0 & \text{hyperbolic}\end{cases}$

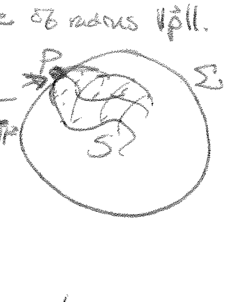
Prop: If S is $\underbrace{\text{cpt. sfc.}}_{\text{closed and bounded}}$, $\exists$ point P of S w/ positive Gaussian curvature.

Extreme Value Thm: If $f: \underset{\text{cpt}}{K} \to \mathbb{R}$, then $\exists\, p, q \in K$ s.t. $f(q) \le f(x) \le f(p)$ $\forall\, x \in K$.

Pf. Let $f(\vec{v}) = \|\vec{v}\|^2$. $\exists\, \underset{\vec{p}}{P} \in$ Sfc attaining max. $\Rightarrow$ S inside sphere of radius $\|\vec{p}\|$.

But the tangent spaces of S, sphere Σ are the same...

Curvature at least $\frac{1}{\|\vec{p}\|}$



$\vec{\gamma}$ unitspeed curve thru P at $t=0$ $\Rightarrow$ $f(\vec{\gamma})$ achieves max at $t=0$

$\Rightarrow \frac{d}{dt} f(\gamma(t)) = 0$, $\frac{d^2}{dt^2}(f(\gamma(t))) \le 0$ $\Rightarrow \frac{d}{dt}\underbrace{\|\gamma\|^2}_{(\gamma\cdot\gamma)} = 0$, $\frac{d^2}{dt^2}\|\gamma\|^2 \le 0$

$\gamma \cdot \dot\gamma = 0$, $2\dot\gamma\cdot\dot\gamma + 2\gamma\cdot\ddot\gamma \le 0$ or $\gamma\cdot\ddot\gamma + 1 \le 0$

So $\vec{p} = \vec{\gamma}(0)$ is $\perp$ tangent plane of S at P.

We have $\vec{N} = \pm\frac{\vec{p}}{\|\vec{p}\|}$. But $\kappa_n = \ddot\gamma(0)\cdot\vec{N} = \pm\frac{\ddot\gamma(0)\cdot\gamma(0)}{\|p\|} \le \frac{-1}{\|p\|}$ so $\kappa_n \le \frac{-1}{\|p\|}$ or $\ge \frac{1}{\|p\|}$, for all curves.

Now Cor 6.2 $\Rightarrow K \ge \frac{1}{\|p\|^2} > 0$ at P. ∎

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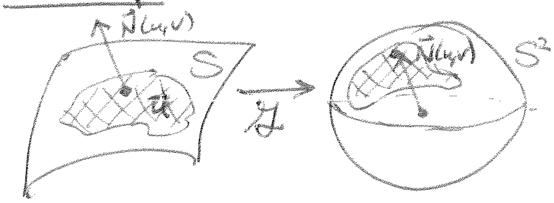
76 Gauss Map



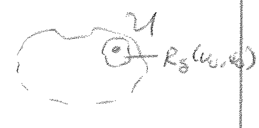
// Gauss Map, can be defined for any orientable surface.

Recall: For $\vec{\gamma}$ unit speed plane curve, $K_S = \frac{d\varphi}{ds}$  for fixed angle φ

Rmk: Let $R \subseteq \mathcal{U}$ (sfc. patch). Then rate of change of $\vec{N}$ per unit area $\approx \frac{\text{area } \vec{N}(R)}{\text{area } \vec{\sigma}(R)} \approx \frac{A_{\vec{N}}(R)}{A_{\vec{\sigma}}(R)}$.

where $A_{\vec{\sigma}}(R) = \iint_R \|\vec{\sigma}_u \times \vec{\sigma}_v\| \, du\, dv$

Theorem: Let $\vec{\sigma}: \mathcal{U} \to \mathbb{R}^3$ be sfc, $(u_0,v_0) \in \mathcal{U}$, $\delta > 0$ s.t. $R_\delta(u_0,v_0) \subset \mathcal{U}$.



Then $\lim_{\delta \to 0} \frac{A_{\vec{N}}(R_\delta)}{A_{\vec{\sigma}}(R_\delta)} = |K|$.

Pf: Ratio is $\approx \frac{\|\vec{N}_u \times \vec{N}_v\|}{\|\vec{\sigma}_u \times \vec{\sigma}_v\|} \Delta u \Delta v$

But $\vec{N}_u \times \vec{N}_v = (a\sigma_u \times b\sigma_v) \times (c\sigma_u \times d\sigma_v) = (ad-bc)\sigma_u \times \sigma_v = \det \mathcal{W}\, \sigma_u \times \sigma_v = K \sigma_u \times \sigma_v$.

Precisely, we have Ratio $= \frac{\iint_{R_\delta} |K| \, \|\sigma_u \times \sigma_v\| du dv}{\iint \|\sigma_u \times \sigma_v\| du dv}$.

But given $\epsilon > 0$, choose $\delta > 0$ s.t. $|K(u,v) - \underset{K_0}{\underset{\|}{K(u_0,v_0)}}| < \epsilon$ for all pts in R_δ.

Then $|K_0| - \epsilon < |K| < |K_0| + \epsilon$

so $\iint_{R_\delta} (|K_0| - \epsilon) \|\sigma_u \times \sigma_v\| du dv < \iint_{R_\delta} |K| \|\sigma_u \times \sigma_v\| du dv < \iint_{R_\delta} (|K_0| + \epsilon) \|\sigma_u \times \sigma_v\| du dv$

so $|K_0| - \epsilon < \text{Ratio} < |K_0| + \epsilon$

Since this is true for all ϵ, $\lim_{\delta \to 0} \text{Ratio} = |K_0|$. ∎ // Can also check the signs!

Ex Application: Plane $\Rightarrow \vec{N}(R)$ has zero area $\Rightarrow K = 0$.
Cylinder $\Rightarrow \vec{N}(R)$ 1-dim'l $\Rightarrow K = 0$.
Sphere $\Rightarrow \vec{N}(R)$ has area dependent only on radius of sphere $\Rightarrow$ curvature $\frac{1}{r^2}$.

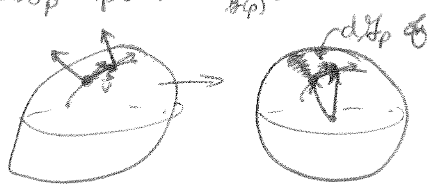
Ex: Torus:  ← part of positive curvature covers the entire sphere exactly once
$\Rightarrow \mathcal{G}$ maps to whole sphere w/ area 4π.

More on the Gauss Map

$\mathcal{W} = d\mathcal{G}$ differential. What is a differential?
→ matrix of partials

$\mathcal{G}: S \to S^2$

$d\mathcal{G}_p : T_p(S) \to T_{\mathcal{G}(p)}(S^2)$ can be taken as a map $T_p(S) \to T_p(S)$ since tangent spaces are parallel.



$d\mathcal{G}_p$ measures how quickly ~~tangent~~ normal vectors diverge!

In local coords, we have $\mathcal{G}(\sigma(u,v)) = \dfrac{\vec{\sigma}_u \times \vec{\sigma}_v}{\|\vec{\sigma}_u \times \vec{\sigma}_v\|}$ ... But $\vec{N}_u = a\vec{\sigma}_u + b\vec{\sigma}_v$
$\vec{N}_v = c\vec{\sigma}_u + d\vec{\sigma}_v$

So for a tangent vector $\binom{u'}{v'}$, we have $d\mathcal{G}\binom{u'}{v'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{u'}{v'} = \mathcal{W}\binom{u'}{v'}$.

Therefore, $\mathcal{W}$ is the differential of the Gauss map $\mathcal{W}_p = d\mathcal{G}_p$.

And we can say $K = \det(d\mathcal{G}_p)$, $H = -\frac{1}{2}\mathrm{tr}(d\mathcal{G}_p)$, κ_i are the eigenvalues of $d\mathcal{G}_p$.