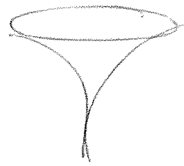
Hyperbolic Space

① 

pseudosphere has constant $_\wedge$Gaussian curvature -1

$\vec{\sigma}(u,v) = \langle e^u \cos v,\ e^u \sin v,\ \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u}) \rangle$ $[u \le 0,\ 0 \le v \le 2\pi]$ ← unit-speed $\dot{f}^2+\dot{g}^2=1$

OR more simply $\vec{\sigma}(u,v) = \langle u\cos v,\ u\sin v,\ \sqrt{1-u^2} - \cosh^{-1}(\frac{1}{u}) \rangle$

We have $E=1, F=0, G=e^{2u}$ and $L=*, M=0, N=e^u(\dot{g})$ $\Rightarrow$ [redacted] $K = -\frac{\ddot{f}}{f} = -1$

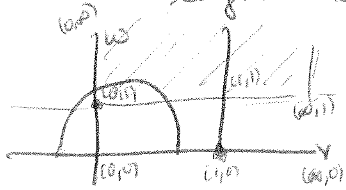
So the $\mathcal{F}_I = du^2 + e^{2u} dv^2$.

If we reparametrize, letting $w = e^{-u}$ $[w \ge 1]$, we have $E = \vec{\sigma}_u \cdot \vec{\sigma}_u = (\vec{\sigma}_w \frac{dw}{du}) \cdot (\vec{\sigma}_w \frac{dw}{du}) = (\vec{\sigma}_w \cdot \vec{\sigma}_w)(\frac{dw}{du})^2$

so $w' = -e^{-u} = -w$ $\qquad = (\vec{\sigma}_w \cdot \vec{\sigma}_w) w^2$

So $E du^2 + G dv^2 \to \frac{1}{w^2} dw^2 + \frac{1}{w^2} dv^2$
$du^2 + e^{2u} dv^2$
$= \frac{1}{w^2}(dv^2 + dw^2) = \frac{1}{w^2} \underbrace{dx^2}_{\text{standard Euclidean}}$

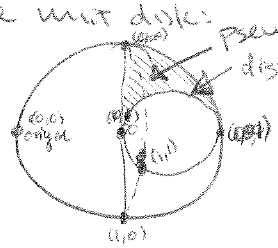
So we have inner product $\langle v,w \rangle = 0$; $\langle v,v \rangle = \frac{1}{w^2}$, $\langle w,w \rangle = \frac{1}{w^2}$.
∴ length $\to \infty$ as $w \to 0$



"Upper Half-Space Model"

If we reparametrize, letting $U = \frac{v^2+w^2-1}{v^2+(w+1)^2}$, $V = \frac{-2v}{v^2+(w+1)^2}$, we get $\frac{dU^2+dV^2}{(1-U^2-V^2)^2}$
$= \frac{1}{(1-r^2)^2} \underbrace{dx^2}_{\text{standard Euclidean}}$

If we ~~use~~ $\frac{4}{(1-r^2)^2} dx^2$, then we get curvature -1 on the unit disk:



$w=1\ v=\infty$
$V=0$

$\frac{-2v}{v^2+4}$

$\frac{v^2}{v^2+4}$

$(1,0)$: $\frac{-2}{1+1} = -1$

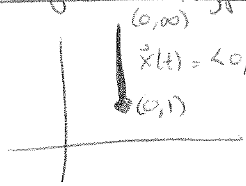
$L = \int \sqrt{ds^2}$ $\qquad x(t) = \langle r\cos t, r\sin t \rangle$

Eg. what is length of circle @ radius r? $\qquad dx^2 = \sqrt{x'^2+y'^2}\,dt = r\,dt$

$\int_{\theta=0}^{2\pi} \frac{1}{(1-r^2)} r\,dt = \boxed{\frac{2\pi r}{(1-r^2)}}$.

On the other hand, @ Eucl. radius r, hypo. length is $\int_0^r \frac{1}{(1-t^2)} dt = \operatorname{arctanh} t \Big|_0^r = \operatorname{arctanh} r$. $\qquad x(t) = \langle t, 0 \rangle \quad dx^2 = 1\,dt$

Length in Hyperbolic Space:



$\vec{x}(t) = \langle 0, t\rangle$, $t = 1 \to N$ $\quad |\vec{x}'(t)| = 1$. So $L = \int_1^N \sqrt{\frac{1}{y(t)^2}}\, |\vec{x}'(t)|\, dt = \int_1^\infty \frac{1}{t}\, dt = \log t \Big|_1^N = \log N$

So e.g. $(0,1) \to (0,e)$ has distance 1
$(0,e) \to (0,e^2)$ has distance 2
⋮
$(0,0) \to (0,1)$ has ∞ distance...
indeed $(0,\frac{1}{e}) \to (0,1)$ has distance 1
$(0,\frac{1}{e^2}) \to (0,\frac{1}{e})$ has distance 1, etc.

However, horizontal lines have the expected lengths.

In the circle, $L = \int \sqrt{ds^2} = \int \frac{2}{1-|x|^2} |x'(t)|\, dt$

Let $x(t) = \langle t, 0\rangle$, $t = 0 \to R$. Then $L = \int_0^R \frac{2}{1-t^2}\, dt = 2 \cdot \frac{1}{2} \ln\left|\frac{1+t}{1-t}\right|_0^R = \ln \frac{1+R}{1-R}$.

Let $x(t) = \langle R\cos t, R\sin t\rangle$, $t = 0 \to 2\pi$. Then $L = \int_0^{2\pi} \frac{2}{1-R^2} R\, dt = \frac{4\pi R}{1-R^2}$.

~~Therefore $C/2\pi r$~~

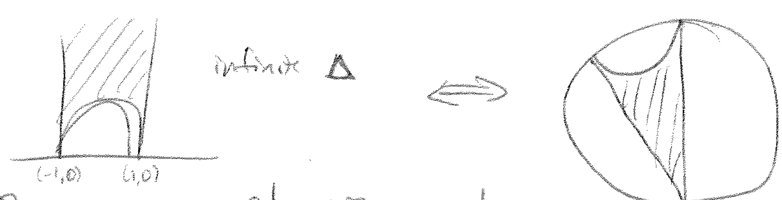
If $r = \ln \frac{1+R}{1-R}$, then $e^r = \frac{1+R}{1-R} \Rightarrow (1-R)e^r = 1+R \Rightarrow e^r - Re^r = 1+R \Rightarrow e^r - 1 = R(1+e^r)$
$\Rightarrow R = \frac{e^r - 1}{e^r + 1}$

So the ratio $C/2\pi r = \frac{2R}{1-R^2} \Big/ \ln\frac{1+R}{1-R} = 2R(1-R^2)^{-1} \ln\left(\frac{1+R}{1-R}\right)^{-1}$

Letting $\Phi(r) = \frac{C(r)}{2\pi r}$, we have $\Phi(r) = \frac{e^r - e^{-r}}{2r} = \frac{\sinh(r)}{r}$

Then $\Phi''(0) = \frac{1}{3}\ldots$ Thm: this is $-\frac{1}{3}$ times the Gaussian curvature!

e.g. Area?



$$\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx = \int_{-1}^{1} -\frac{1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{\cos\theta} \cos\theta\, d\theta = \pi.$$

Theorem: Area of hyperbolic Δ is $\pi - \sum$ interior angles

TOPS 35500