26/1 **Goal:** find $y(x)$ to minimize $\int_a^b F(x,y,y')dx$ for some continuous F.

Thm: Min/Max occur when $\boxed{\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}}$ (Euler's Eqn.)

Pf: Let $y(x) \to \underset{\text{where } \eta(a)=\eta(b)=0 \text{ @ endpts.}}{y(x)+\epsilon\eta(x)}$, so $C(\epsilon) = \int_a^b F(x, y+\epsilon\eta, y'+\epsilon\eta')dx$

Then $C'(0) = 0 = \frac{d}{d\epsilon}\left(\int_a^b F(x,y+\epsilon\eta,y'+\epsilon\eta')dx\right)\Big|_{\epsilon=0} = \int_a^b \frac{\partial F}{\partial \epsilon}dx = \int_a^b \frac{\partial F}{\partial x}\overset{0}{\cancel{\frac{\partial x}{\partial \epsilon}}} + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' dx$

$$= \int_a^b \frac{\partial F}{\partial y}\eta - \left(\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta\, dx = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta\, dx$$

Holds ∀ smooth η, hence $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$. ■

Ex: ① Maximize $\int_{[illegible]}^2 \frac{y'}{x}dx$ w/ $y([illegible]) = [illegible]$, $y(2)=2$.

$F = \frac{y'}{x}$ $F_y = 0$, $F_{y'} = \frac{1}{x}$, so $-\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x^2} = 0 \iff$ impossible $\iff$ no local extrema

② Maximize $\int_1^2 \frac{y'^2}{x}dx$.

$F = \frac{y'^2}{x}$ $F_y = 0$ $F_{y'} = \frac{2y'}{x} = C_1$ $y' = C_1 x$ $y = C_1 x^2 + B \iff y = \frac{1}{3}x^2 + \frac{2}{3}$

27/1 **Thm:** ① If $\frac{\partial F}{\partial y} = 0$, so $\int_a^b F(x,y')dx$, then max/min when $\frac{\partial F}{\partial y'} = C_1$. **Pf:** $\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$. ■

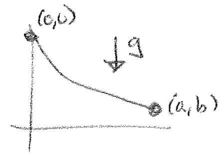
② If $\frac{\partial F}{\partial x} = 0$, so $\int_a^b F(y,y')dx$, then $F - \frac{\partial F}{\partial y'}y' = C_1$.

Pf: $\frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right)$

$$= \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}\frac{dy'}{dx} - \frac{\partial F}{\partial y'}\frac{dy'}{dx} - \frac{d}{dx}\frac{\partial F}{\partial y'}y'$$

$$= \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)y' = 0.$$ ■

Ex: Brachistochrone:



$$\frac{1}{2}mv^2 = mgy \iff v = \sqrt{2gy}$$

Let s = arc length, t = time $\Rightarrow \frac{ds}{dt} = v = \sqrt{2gy}$, $ds = \sqrt{1+y'^2}dx$, so $dt = \sqrt{\frac{1+y'^2}{2gy}}dx$

So T = time to fall $= \int_0^a \sqrt{\frac{1+y'^2}{2gy}}dX$.

1st Integral: $\sqrt{\frac{1+y'^2}{2gy}} - \frac{1}{2}\left(\frac{1+y'^2}{2gy}\right)^{-1/2}\frac{2y'y'}{2gy} = C_1 \iff \frac{1+y'^2}{2gy} - \frac{1}{2}\left(\frac{y'^2}{gy}\right) = C_1\sqrt{\frac{1+y'^2}{y}} \iff \overbrace{(1+y'^2)y - yy'^2}^{1} = C_1\sqrt{y(1+y'^2)}$

$\iff y'^2 = \frac{c}{y} - 1 \iff \frac{dy}{dx} = \sqrt{\frac{c-y}{y}}$

From $y' = \sqrt{\frac{c-y}{y}}$, let $y = c\sin^2\theta \Rightarrow y' = \sqrt{\frac{c(1-\sin^2\theta)}{c\sin^2\theta}} = 2c\sin\theta\cos\theta\,\theta'$
$= \frac{\cos\theta}{\sin\theta} \iff \theta' = \frac{c}{\sin^2\theta} \leadsto \boxed{x = c(\varphi - \sin\varphi),\ y = c(1-\cos\varphi)}$

<u>cycloid</u>

2-Feb 07

1870 Weierstrass Calc. Variations

Isoperimetric Ineq:

Thm: For γ simple closed curve, $A(\text{int}(\gamma)) \leq \frac{1}{4\pi} \ell(\gamma)^2$, w/ eq. only for circles

Pressley Pf:

Lem: $F:[0,\pi] \to \mathbb{R}$ smooth, $F(0)=F(\pi)=0$, then $\int_0^\pi (\dot{F})^2 dt \geq \int_0^\pi F^2 dt$, w/ eq. only for $F = A\sin t$.

Pf: ① If $F = G\sin t$, then
$$\int_0^\pi (\dot{F}^2 - F^2)dt = \int_0^\pi (\dot{G}^2 \sin^2 t + G^2\cos^2 t + 2G\dot{G}\sin t\cos t - G^2\sin^2 t)dt$$
$$= \int_0^\pi \dot{G}^2 \sin^2 t\,dt + \int_0^\pi G^2(\cos^2 t - \sin^2 t) + 2G\dot{G}(\sin t\cos t)\,dt$$
$$= \int_0^\pi \dot{G}^2 \sin^2 t\,dt + \int_0^\pi \frac{d}{dt}\left[\overset{0}{\cancel{G^2(\sin t\cos t)}}\right]dt$$
$$= \int_0^\pi (\dot{G}\sin t)^2 dt \geq 0. \quad ✓$$

② **Eq?** Zero? Then $\dot{G}\sin t = 0\ \forall t$, so $\dot{G} = 0$ for all $t \in (0,\pi)$

so $G = C_1$ and $F = C_1 \sin t$. ∎

Pf: ① Assume γ has period π, parm. by $t = \frac{\pi s}{\ell(\gamma)}$, w/ $\gamma(0) = \gamma(\pi) = 0$.

$A = \iint r\,dr\,d\theta$

Then $\|\dot{r}\|^2 = \dot{x}^2 + \dot{y}^2 = \left(\frac{\ell(\gamma)}{\pi}\right)^2 = \frac{\ell(\gamma)^2}{\pi^2}$.

② In polar coords, $A(\text{int}(\gamma)) = \frac{1}{2}\int_0^\pi (x\dot{y} - y\dot{x})dt = \frac{1}{2}\int_0^\pi r^2 d\theta = \frac{1}{2}\int_0^\pi r^2\dot{\theta}\,dt$

and $\ell(\gamma) = \int_0^\pi (\dot{x}^2 + \dot{y}^2)dt = \int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2)dt = \frac{\ell(\gamma)^2}{\pi}$.

③ Thus, $\frac{\ell(\gamma)^2}{4\pi} - A(\text{int}(\gamma)) = \frac{1}{4}\int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta})dt = \frac{1}{4}\int_0^\pi \left(r^2(\dot{\theta}-1)^2 + (\dot{r}^2 - r^2)\right)dt \geq 0$

④ **Eq?** Zero $\Leftrightarrow r^2(\dot{\theta}-1)^2 = 0$, $\dot{r}^2 - r^2 = 0$, i.e. $\dot{\theta} = 1 \Rightarrow \theta = t + C_1$ and $r = A\sin t$ by above

$\Leftrightarrow r = A\sin(\theta - C_1) \Leftrightarrow$ circle. ∎

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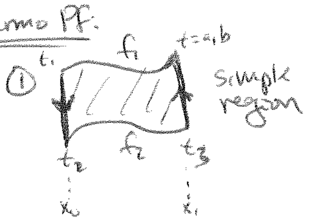
doCarmo Pf.

①

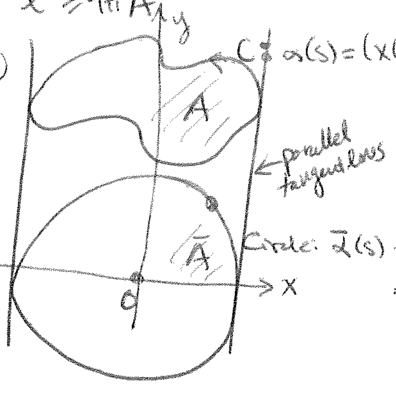


$$A = \int_{x_0}^{x_1} (f_1 - f_2)dx = \int_a^{t_1} y\dot{x}\,dt - \int_{t_2}^{t_3} y\dot{x}\,dt = -\int_a^b y\dot{x}\,dt$$

$\Rightarrow$ Green's Thm $A = \frac{1}{2}\int_a^b (xy' - yx')dt = \int_a^b xy\,dt = -\int_a^b yx'dt$

Thm: $\ell^2 \geq 4\pi A$

Pf: ①



C: $\alpha(s) = (x(s), y(s))$, $s \in [0,\ell]$

Circle: $\bar{\alpha}(s) = (\bar{x}(s), \bar{y}(s))$
$= (x(s), \bar{y}(s))$, $s \in [0,\ell]$

② $A = \int_0^\ell xy'ds$, $\bar{A} = \pi r^2 = -\int_0^\ell \bar{y}x'ds$

$\Downarrow$

$$A + \pi r^2 = \int_0^\ell (xy' - \bar{y}x')ds$$
$$\leq \int_0^\ell |xy' - \bar{y}x'|ds$$
$$\leq \int_0^\ell \sqrt{(x^2+\bar{y}^2)(x'^2+y'^2)}ds = \int_0^\ell \sqrt{x^2+\bar{y}^2}ds = \int_0^\ell r\,ds = \ell r$$

So $\sqrt{A}\sqrt{\pi r^2} \leq \frac{1}{2}(A + \pi r^2) \leq \frac{1}{2}\ell r$ (AM-GM)

$\Leftrightarrow 4\pi A r^2 \leq \ell^2 r^2$ ✓