27-Feb

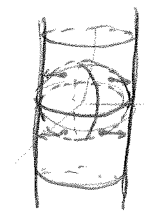
§5.5 - Equiareal Maps

Recall: Given E, F, G:
1. $A.L. = \int \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}$
2. ~~Isometry~~ $\Leftrightarrow$ diffeo f preserves E, F, G.
3. Conformal $\Leftrightarrow$ diffeo. f takes E, F, G to $\lambda E, \lambda F, \lambda G$.
4. $S.A. = \iint_R \sqrt{EG - F^2}\, du\,dv$.

Def: Equiareal $\Leftrightarrow$ diffeo f takes regions to regions of equal area.

5. Equiareal $\Leftrightarrow$ $EG - F^2$ is the same/preserved under f.

Thm (Archimedes): The map f is equiareal, as described @ right.



f maps pts on unit sphere to closest point on cylinder

$f = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$

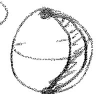
Pf: For sphere, use param. $\sigma(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$

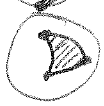
Then $f \circ \sigma = (\cos\varphi, \sin\varphi, \sin\theta)$

For σ, $E = 1$, $F = 0$, $G = \cos^2\theta$

For $f\circ\sigma$, $E = \cos^2\theta$, $F = 0$, $G = 1$

Therefore, $EG - F^2$ is preserved. ∎

Appl: (1)  Area btwn 2 great circles $= 2\theta$ via this equiareal map.

(2)  Area of $\triangle$ w/ angles $\theta_1, \theta_2, \theta_3$ is $\theta_1 + \theta_2 + \theta_3 - \pi$.

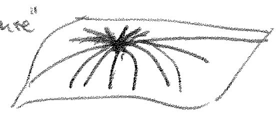
Curvature

Recall: For ~~plane~~ curves, $\kappa = \|\ddot{\gamma}\|$, for γ a unit-speed curve. More generally $\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$

A circle of radius R has constant curvature $\frac{1}{R}$ $\Rightarrow$ "osculating circle"

How do we extend this idea to surfaces?

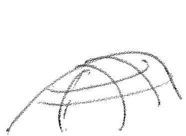
1. "Sectional Curvature"

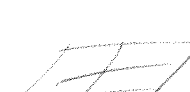


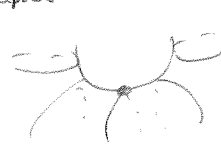
$\leftarrow$ each point on sfc has lots of geodesic curves through it. We can look at the curvature of any of these. (But these should be geodesics!!)

3. "Mean Curvature": take average of the ~~two~~ 2
2. "Principal Curvatures": the max/min curvatures
4. "Gaussian curvature": the product of the two principal curvatures

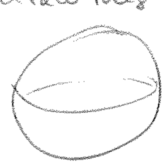
Possibilities:

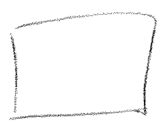
 pos. curv.

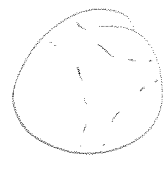
 zero curv.

 neg. curv.

One way to draw these:

 constant $K=1$

 constant $K=0$

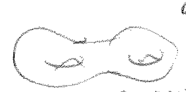
 constant $K=-1$

(conformal, but distances are wrong!!)

$ds^2 = \frac{4}{(1-r^2)^2}\, dx^2$.

sphere 

 torus

 ....

everything else.

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