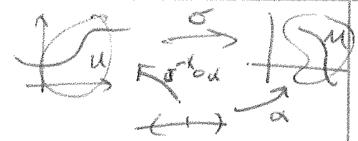
Tangent Space



Tangent Spaces

We identified for curves $\alpha:(-\epsilon,\epsilon)\to M$, local patch $\sigma: U \to M$,
the "tangent vector" with $(x_1'(0),\dots,x_n'(0))$ where $\sigma^{-1}\circ\alpha(t)=(x_1(t),\dots,x_n(t))$
Ie. Calculus is defined on the domain!
- requires careful approach to "gluing"... is it independent of patch??

Given $f: M\to\mathbb{R}$ differentiable, we have $\frac{d}{dt}(f\circ\alpha) = \left(\sum_i x_i'(0)\frac{\partial f}{\partial x_i}\right)$ by the chain rule (directional derivative)

Therefore, makes sense to define the tangent vector to the curve α at $t=0$ to be a fn. $\alpha'(0): \mathcal{D}\to\mathbb{R}$, where $\mathcal{D}$ = spc. diff'ble fns on M at p.

$$\alpha'(0)f \equiv \left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}$$

Since on a patch σ, we can represent this by $\left(\sum_i x_i'(0)\frac{\partial}{\partial x_i}\right)f$, it makes sense to ID $\alpha'(0) \leftrightarrow \sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0$

Hence, the tangent vectors are in the span of $\left\{\left(\frac{\partial}{\partial x_i}\right)_0\right\}$.

In previous work, these are the tangents to curves along the coordinate lines.
- the space is called T_pM.
- choice of local coords determines assoc. basis, but the space itself is independent of this.

The differential $d\varphi_p$ is the unique map $d\varphi_p: T_pM_1 \to T_{\varphi(p)}M_2$ between tgt. vectors given $\varphi: M_1\to M_2$.

If $\varphi: M_1\to M_2$ is differentiable, bijective, φ^{-1} differentiable, then it is a diffeomorphism
$\Rightarrow d\varphi_p$ is an isomorphism.

Forms

Ex: Surfaces $\Rightarrow T_pM$ has basis $\left\{\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right\}$ for $\vec\sigma(u,v)$.

A differential form is a differentiable linear fn $T_pM\to\mathbb{R}$
By "duality" they have a "dual basis" $du: \frac{\partial}{\partial u}\mapsto 1,\ \frac{\partial}{\partial v}\mapsto 0$ and $dv: \frac{\partial}{\partial u}\mapsto 0,\ \frac{\partial}{\partial v}\mapsto 1$. T_p^*M = cotangent space

An n-form is a "power" of lower-order forms, maps n tangent vectors to $\mathbb{R}$.

So ds^2 is a 2-form .... $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$.
It maps $\frac{\partial}{\partial u},\frac{\partial}{\partial u}\mapsto E$ $\quad\frac{\partial}{\partial u},\frac{\partial}{\partial v}\mapsto 2F$ $\quad\frac{\partial}{\partial v},\frac{\partial}{\partial v}\mapsto G$ via dot product

$(EG-F^2)^{1/2}\,du\,dv$ is the area form, also written $\|\vec\sigma_u\times\vec\sigma_v\|\,du\,dv$.

Vector Fields

Vector field: choose tgt. vector @ each pt in a differentiable way.
$\Longleftrightarrow$ takes each $p\in M$ to some $X(p)\in T_pM$
$\Longleftrightarrow$ a "differentiable section" of the tangent bundle $TM = M\times T_pM$
a map $M\to TM$

Generalizes idea of dir'l derivative. $(Xf)(p) = \sum_i a_i(p)\frac{\partial f}{\partial x_i}(p)$ where $X(p)=\sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}$. (a functions)

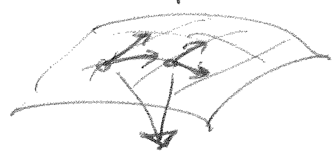
Riem. metrics

Riemannian Metrics

Given diff. mfd M, a Riemannian metric is an inner product $\langle , \rangle_p$ defined on the tangent space T_pM at all points such that

$g_{ij} = \langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q$ is a differentiable fn. on U.

e.g. $E, 2F, G$ part of g_{ij}



$\mathbb{R}$

Pairs of vectors

$\downarrow$ differentiable for fixed choice of basis vectors

$\mathbb{R}$

Ex: If M is in Riem. space, $\langle , \rangle_p$ is the usual inner product.

* These exist $\forall$ diff'ble manifolds

Isometries $\Longleftrightarrow$ $\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$ always.

Then: A curve $c: [a,b] \to M$ has $\boxed{l_a^b(c) = \int_a^b \langle \frac{dc}{dt}, \frac{dc}{dt} \rangle^{1/2} dt.}$

A region $R \subset M$ w/ compact closure has volume $\int_{x^{-1}(R)} \sqrt{\det(g_{ij})}\, dx_1 \ldots dx_n$

$\downarrow$ Ex:

$\int \sqrt{EG - F^2}\, du\, dv$

Hyper. Geom.