# MA489 - Chapter 7.1 - 7.3

Recall: $K = K_1 K_2$ Gaussian cvtr $= \det \mathcal{W} = \dfrac{LN - M^2}{EG - F^2}$

$H = \frac{1}{2}(K_1 + K_2)$ mean cvtr $= \frac{1}{2} \operatorname{tr} \mathcal{W} = \frac{1}{2}\left(\dfrac{LG - 2MF - NE}{EG - F^2}\right)$

7.2 Pseudosphere: sfc. revolution $\vec{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$.

Recall: (p. 125) Ex 6.2

Assume $f(u) > 0$ and $u \mapsto (f(u), 0, g(u))$ is unit-speed. $\Rightarrow \dot{f}^2 + \dot{g}^2 = 1$

Let $\dot{}$ denote $d/du$.

Then $\vec{\sigma}_u = \langle \dot{f}\cos v, \dot{f}\sin v, \dot{g}\rangle$, $\vec{\sigma}_v = \langle -f\sin v, f\cos v, 0\rangle$

$\vec{\sigma}_{uu} = \langle \ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}\rangle$, $\vec{\sigma}_{vv} = \langle -f\cos v, -f\sin v, 0\rangle$

$\vec{\sigma}_{uv} = \langle -\dot{f}\sin v, \dot{f}\cos v, 0\rangle$

So $E = \|\vec{\sigma}_u\|^2 = \dot{f}^2 + \dot{g}^2 = 1$, $G = \|\vec{\sigma}_v\|^2 = f^2$, $F = \vec{\sigma}_u \cdot \vec{\sigma}_v = \emptyset$

$$\boxed{\mathcal{F}_I = du^2 + f^2 dv^2}$$

$\vec{\sigma}_u \times \vec{\sigma}_v = \langle -f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}\rangle$

$\|\vec{\sigma}_u \times \vec{\sigma}_v\|^2 = (f\dot{g})^2 + (f\dot{f})^2 = f^2$

$\vec{N} = \dfrac{\vec{\sigma}_u \times \vec{\sigma}_v}{\|\vec{\sigma}_u \times \vec{\sigma}_v\|} = \langle -\dot{g}\cos v, -\dot{g}\sin v, \dot{f}\rangle$

So $L = \vec{\sigma}_{uu} \cdot \vec{N} = -\ddot{f}\dot{g} + \dot{f}\ddot{g}$ $\quad M = \vec{\sigma}_{uv} \cdot \vec{N} = \dot{f}\dot{g}\sin v\cos v - \dot{f}\dot{g}\sin v\cos v = 0$

$N = \vec{\sigma}_{vv} \cdot \vec{N} = f\dot{g}$

$$\boxed{\mathcal{F}_{II} = (\dot{f}\ddot{g} - \ddot{f}\dot{g})\,du^2 + (f\dot{g})\,dv^2}$$

Therefore, $K = \dfrac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2}$, $H = \dfrac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f^2 + f\dot{g}}{f^2}$ $\quad$ But $\dot{f}^2 + \dot{g}^2 = 1$

$\Rightarrow \dot{f}\ddot{f} + \dot{g}\ddot{g} = 0 \Rightarrow \dot{g}\ddot{g} = -\dot{f}\ddot{f}$

$= \dfrac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g}}{f}$ $\quad = \dot{f}\ddot{g} - \ddot{f}\dot{g} + \dfrac{\dot{g}}{f} = \dfrac{-\ddot{f}}{\dot{g}} + \dfrac{\dot{g}}{f}$

$\dot{g}\dot{f}\ddot{g} - \ddot{f}\dot{g}\dot{g} = -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 = -\ddot{f}(\dot{f}^2 + \dot{g}^2) = -\ddot{f}$

So $\boxed{K = -\dfrac{\ddot{f}}{f}}$

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7.2 What if $K = 0$? Then $\ddot{f} = 0 \Rightarrow f = au + b \Rightarrow$ part of cone or cylinder or annulus

$\vec{\sigma}(u,v) = \langle b\cos v, b\sin v, 0\rangle + u\langle a\cos v, a\sin v, \sqrt{1-a^2}\rangle$.

What if $K = 1$? Then $\ddot{f} + f = 0 \Rightarrow f(u) = a\cos(u+b) \rightsquigarrow a\cos(u)$ by simplifying/reparametrizing

$\Rightarrow g(u) = \int \sqrt{1 - a^2\sin^2 u}\, du$

If $a = 0$: $g(u) = \int du = u$ .... $\vec{\sigma} = \langle 0, 0, u\rangle$ not sfc.

If $a = 1$: $g(u) = \int \cos u\, du = \sin u$ $\Rightarrow \vec{\sigma} = \langle \cos u\cos v, \cos u\sin v, \sin u\rangle$ sphere!

What if $K = -1$? $\ddot{f} = -f$ ⇒ $f = ae^u + be^{-u}$

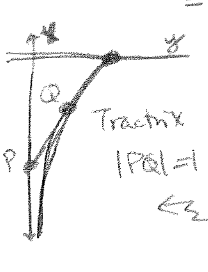
Assuming $f(u) = e^u$ ... $g(u) = \int \sqrt{1 - e^{2u}}\, du = \int \dfrac{\sqrt{1-v^2}}{v} dv = \sqrt{1-v^2} + \int \dfrac{dv}{v\sqrt{1-v^2}}$

$= \sqrt{1 - e^{2u}} - \int \dfrac{dw}{\sqrt{w^2 - 1}}$

$= \sqrt{1 - e^{2u}} - \cosh^{-1} w = \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u})$

So curve is $\langle e^u, \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u})\rangle$ for $u < 0$.

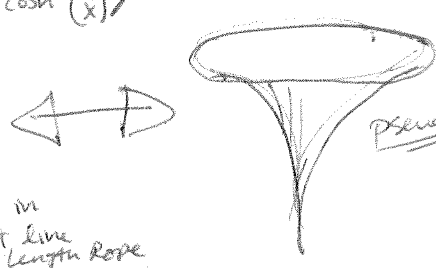
$= \left\langle x, \sqrt{1 - x^2} - \cosh^{-1}\left(\tfrac{1}{x}\right)\right\rangle$




Tractrix

$|PQ| = 1$ always!

⟷ Drag Box in straight line for fixed length Rope



Pseudosphere.

7.3 Flat Surfaces

Flat Sfc $\Leftrightarrow$ $K=0$ everywhere

$\Longrightarrow$ Pf. relies on inv. fn. thm.

Prop[72]: If not @ umbilic pt, can reparametrize so that $F=M=0$ at a given point.
$\Longleftrightarrow$ principal vectors are $\vec{\sigma}_u, \vec{\sigma}_v$ w/ principal curvatures $\frac{L}{E}, \frac{N}{G}$. so $K = \frac{LN}{EG}$

Prop[73]: Let P be pt of flat sfc. not umbilic. Then $\exists$ patch containing P that is a ruled sfc.

Pf: ① Can assume $L \neq 0$, $N=0$ everywhere on the patch $\Rightarrow$ $\mathcal{F}_{II} = L\,du^2$.
② Can show consequently that $u$=constant curves are lines
bcs. unit tangents are $\vec{t} = \vec{\sigma}_v/\sqrt{G}$ and can show $\vec{t}_v = 0$. ∎

Ruled Sfcs: $K=0 \Leftrightarrow \dot{\vec{\delta}} \cdot (\vec{\sigma}_u \times \vec{\sigma}_v) = 0$ everywhere... $\vec{\sigma}(u,v) = \vec{\gamma}(u) + v\,\vec{\delta}(u)$.
$\Longleftrightarrow \dot{\vec{\delta}} \cdot (\dot{\vec{\gamma}} \times \vec{\delta}) = 0 \Longleftrightarrow \dot{\delta}, \dot{\gamma}, \delta$ everywhere linearly dependent

From this we get $\|\delta\| = 1$, $\dot{\delta} = 0 \Rightarrow \delta$ constant, generalized cylinder
$\dot{\delta}$ never zero, $\dot{\gamma} = f\delta + g\dot{\delta}$, if $f = \dot{g}$ everywhere, generalized cone
if $\dot{\delta}$, $f - \dot{g}$ both nowhere zero, tangent developable

see p.103

$\sigma = \gamma + v\dot{\gamma}$