Ex 5.4: $\tilde{E} = \tilde{\sigma}_{\tilde{u}} \cdot \tilde{\sigma}_{\tilde{u}} = \frac{\partial \tilde{\sigma}}{\partial \tilde{u}} \cdot \frac{\partial \tilde{\sigma}}{\partial \tilde{u}} = \left(\frac{\partial u}{\partial \tilde{u}}\sigma_u + \frac{\partial v}{\partial \tilde{u}}\sigma_v\right) \cdot \left(\frac{\partial u}{\partial \tilde{u}}\sigma_u + \frac{\partial v}{\partial \tilde{u}}\sigma_v\right)$

$= \left[\begin{pmatrix} u_{\tilde{u}} & v_{\tilde{u}} \end{pmatrix}\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix}\right]^T \begin{pmatrix} u_{\tilde{u}} & v_{\tilde{u}} \end{pmatrix}\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix} = \begin{pmatrix}\sigma_u & \sigma_v\end{pmatrix}\begin{pmatrix}u_{\tilde{u}} \\ v_{\tilde{u}}\end{pmatrix}\begin{pmatrix} u_{\tilde{u}} & v_{\tilde{u}} \end{pmatrix}\begin{pmatrix}\sigma_u \\ \sigma_v\end{pmatrix}$ ?? works out...

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§5.3 Conformal Maps

Given paths $\gamma$ and $\tilde{\gamma}$, $=\sigma(u,v)$, $=\sigma(\tilde{u},\tilde{v})$ intersecting at $\gamma(t_0) = \tilde{\gamma}(\tilde{t}_0)$, angle is given by $\cos\theta = \frac{\dot{\gamma} \cdot \dot{\tilde{\gamma}}}{\|\dot{\gamma}\|\|\dot{\tilde{\gamma}}\|}$

We have $\dot{\gamma} = \vec{\sigma}_u \dot{u} + \vec{\sigma}_v \dot{v}$, $\dot{\tilde{\gamma}} = \sigma_u \dot{\tilde{u}} + \sigma_v \dot{\tilde{v}}$, so

$\dot{\gamma} \cdot \dot{\gamma} = (\vec{\sigma}_u \cdot \vec{\sigma}_u)\dot{u}^2 + 2(\vec{\sigma}_u \cdot \vec{\sigma}_v)\dot{u}\dot{v} + (\vec{\sigma}_v \cdot \vec{\sigma}_v)\dot{v}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$

$\dot{\gamma} \cdot \dot{\tilde{\gamma}} = (\sigma_u \cdot \sigma_u)\dot{u}\dot{\tilde{u}} + (\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v})(\sigma_u \cdot \sigma_v) + (\dot{v}\dot{\tilde{v}})(\sigma_v \cdot \sigma_v) = E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}$

So $\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}$

"Coordinate Curves": $\gamma(t) = \sigma(a,t)$, $\tilde{\gamma}(t) = \sigma(t,b)$ $\Rightarrow \cos\theta = \frac{F}{\sqrt{EG}} \Rightarrow$ (orthogonal $\Leftrightarrow F = 0$)
$\dot{u} = 0, \dot{v} = 1$ $\quad \dot{\tilde{u}} = 1, \dot{\tilde{v}} = 0$

Def: $f: S_1 \to S_2$ diffeo is conformal iff $\angle(\gamma_1, \tilde{\gamma}_1) = \angle(f\circ\gamma_1, f\circ\tilde{\gamma}_1)$

$f: U \to \mathbb{R}^3$ sfc. patch may be a conformal parm. or a conformal surface patch.

Thm: Diffeo $f: S_1 \to S_2$ conformal $\Leftrightarrow$ $\sigma_1$ and $f\circ\sigma_1$ always have proportional surface patches.

Pf: "$\Leftarrow$" by above equation, the proportionality constant cancels.
"$\Rightarrow$" conversely, can pick specific curves to demonstrate proportionality. ∎

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§5.4 Surface Area:

Area of small piece of surface $\approx \|\sigma_u \times \sigma_v\| \Delta u \Delta v$

Def: Area $A_\sigma(R)$ for $\sigma: U \to \mathbb{R}^3$, $R \subset U$, is $A_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\| \, du\, dv$.

Prop: $\|\sigma_u \times \sigma_v\| = (EG - F^2)^{1/2}$.

$\Rightarrow A_\sigma(R) = \iint_R (EG - F^2)^{1/2} \, du\, dv = \iint_R dA_\sigma$

Prop: $A_\sigma(R)$ indep. of parametrization

Pf: $\iint_R \|\sigma_u \times \sigma_v\| \, du\, dv = \iint_R \left|\det(J(\Phi)^{-1})\right| \|\sigma_u \times \sigma_v\| \, du\, dv = \iint_{\tilde{R}} \|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\| \, d\tilde{u}\, d\tilde{v}$. ∎

Ex: Paraboloid $z = x^2 + y^2$, $z \leq 1$ ??

Parm: $\langle u, v, u^2 + v^2\rangle$ $\quad \sigma_u = \langle 1, 0, 2u\rangle$ $\quad \sigma_v = \langle 0, 1, 2v\rangle$
$E = 1 + 4u^2$ $\quad G = 1 + 4v^2$ $\quad F = 4uv$ $\Rightarrow \iint_R (1 + 4u^2 + 4v^2 + 16u^2v^2 - 16u^2v^2)^{1/2} \, du\, dv$
$= \iint_R (1 + 4u^2 + 4v^2)^{1/2} \, du\, dv$
$= \dots$ (switch to polar)

Hemisphere $\langle u, v, \sqrt{1 - u^2 - v^2}\rangle$ $\quad$ ~~$\sigma_u = \langle 1, 0, 2u\rangle$~~ $\quad$ ~~$\sigma_v = \langle 0, 1, 2v\rangle$~~
~~$E = 1 + 4u^2$, $G = 1 + 4v^2$, $F = 4uv$~~
??