Review of Curvature Definitions — 24-Mar-09

Ch. 6

Review Given path $\vec{\sigma}(u,v)$:

$$\mathcal{F}_I = \underbrace{\|\vec{\sigma}_u\|^2}_{E} du^2 + \underbrace{2\vec{\sigma}_u\cdot\vec{\sigma}_v}_{2F}\, du\,dv + \underbrace{\|\vec{\sigma}_v\|^2}_{G} dv^2 \qquad \text{(arc length)}$$

$$\mathcal{F}_{II} = \underbrace{(\vec{\sigma}_{uu}\cdot\vec{N})}_{L} du^2 + \underbrace{2(\vec{\sigma}_{uv}\cdot\vec{N})}_{2M}\, du\,dv + \underbrace{(\vec{\sigma}_{vv}\cdot\vec{N})}_{N} dv^2 \qquad \text{(curvature)}$$

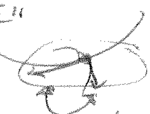
For a unit-speed curve $\gamma(t) = \vec{\sigma}(u(t), v(t))$: $\kappa^2 = \kappa_n^2 + \kappa_g^2$ where $\kappa_n = \ddot{\gamma}\cdot\vec{N}$ normal curvature, $\kappa_g = \ddot{\gamma}\cdot(\vec{N}\times\dot{\gamma})$ geodesic curvature

Then $\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2$ is the normal curvature!

- Normal Curvature relates to the curvature of the sfc. in space
- Geodesic Curvature relates to the curvature on the sfc.
- Principal curvatures are the max + min. normal curvatures

the "Weingarten Matrix" $\mathcal{W}$: $\begin{pmatrix}\vec{N}_u\\ \vec{N}_v\end{pmatrix} = \mathcal{W}\begin{pmatrix}\vec{\sigma}_u\\ \vec{\sigma}_v\end{pmatrix}$ determines how normals change relative to surface coordinates.

Princ. Curvatures computed via $\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0$, or evals. of $\mathcal{F}_I^{-1}\mathcal{F}_{II}$.
They are real numbers and
- $\mathcal{F}_{II} = \kappa\mathcal{F}_I$ implies $\kappa_1 = \kappa_2 = \kappa$, all dir's are principal. "umbilic point"
- If $\kappa_1 \neq \kappa_2$ the princ. directions are orthogonal



principal vectors are the dir'ns corresponding to max/min curvatures

Euler's Thm: $\kappa_n = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta$, where $\theta = \angle(\dot{\gamma}, \vec{t}_1)$ ($\vec{t}_1$ first princ. direction)

Props: Have 4 cases
- $\kappa_1\kappa_2 > 0$ both pos. or neg. "elliptic" resembles elliptic paraboloid
- $\kappa_1\kappa_2 < 0$ "hyperbolic" resembles hyperbolic paraboloid
- $\kappa_1\kappa_2 = 0$, one $\neq 0$ "parabolic" resembles parabolic cylinder
- $\kappa_1 = \kappa_2 = 0$ "planar" resembles plane

behavior up to 2nd order

Ex: Every pt umbilic $\Rightarrow$ part of plane or sphere.

Ch. 7

7.1

Def: Gaussian cvtr $= \kappa_1\kappa_2 = K$
mean cvtr $= \frac{1}{2}(\kappa_1 + \kappa_2) = H$
κ_1, κ_2 are principal curvatures

Props: (i) $K = \dfrac{LN - M^2}{EG - F^2} = \dfrac{\det\mathcal{F}_{II}}{\det\mathcal{F}_I} = \det\mathcal{W}$, (ii) $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)} = \frac{1}{2}\operatorname{tr}(\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \frac{1}{2}\operatorname{tr}(\mathcal{W})$

(iii) principal curvatures are $H \pm \sqrt{H^2 - K}$

Ex: Unit sphere $\kappa_1 = \kappa_2 = 1 \Rightarrow K = H = 1$
Circ. cylinder $\kappa_1 = 1, \kappa_2 = 0 \Rightarrow H = \frac{1}{2}, K = 0$
Sfc. revol. $\mathcal{F}_I = du^2 + f^2 dv^2$, $\mathcal{F}_{II} = (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + (f\dot{g})dv^2 \Rightarrow K = \dfrac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2} = \dfrac{-\ddot{f}}{f}$.