§5 (Pressley)

§5.1 $\vec{\sigma}$ sfc patch $\Rightarrow$ arc length $s = \int_{t_0}^{t} \|\dot{\gamma}\|\, du = \int_{t_0}^{t} (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2} dt$, some E, F, G

$= \int \sqrt{ds^2}$ where $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$. 1st Fundamental Form

Computing these: $E = \|\vec{\sigma}_u\|^2$, $G = \|\vec{\sigma}_v\|^2$, $F = \vec{\sigma}_u \cdot \vec{\sigma}_v$

$$\text{So } ds^2 = (du\ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = (du\ dv)\begin{pmatrix} \vec{\sigma}_u^{\,T} \\ \vec{\sigma}_v^{\,T} \end{pmatrix}(\vec{\sigma}_u\ \vec{\sigma}_v)\begin{pmatrix} du \\ dv \end{pmatrix}.$$

Can use this to compute arc length. Ex: 1,2,3

§5.2 Def: Isometry: diffeomorphism $f: S_1 \to S_2$ taking curves to curves of same length. (preserves sfc. geometry)

Thm: Isometry $\Leftrightarrow$ $f: S_1 \to S_2$ implies σ patch and $f \circ \sigma$ have same 1FF.

Pf: Same 1FF $\Rightarrow$ $\gamma(t)$ and $f \circ \gamma(t)$ have same lengths since both filter thru u, v. ✓

Isometry $\Rightarrow E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2 = E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2$ for all paths $(u(t), v(t))$

If $u = u_0 + t - t_0, v = v_0$ then we get $E_1 = E_2$. Similar for the other two coeffs. ∎

$\therefore$ Length-Preserving $\Leftrightarrow$ 1FF Preserving.

Ex: 5,8

What is a differential form?

"a smooth section of the k^{th} exterior power of the cotangent bundle of the manifold"

→ Provides multilinear map from k^{th} Cartesian power of the tgt spc. at p to $\mathbb{R}$.

(So assigns real # to a k-tuple of tgt. vectors)

Mathind Gift: (possible material)
III-1: "Birth of Manifold"
I-1-3: "Curvature/Gauss-Bonnet"
I-2-3: "3-Mfds"

Eg. $f \mapsto D_u f$ directional derivative

$D_u f = Df \circ u = f_x u_1 + f_y u_2$

$f \mapsto \frac{\partial f}{\partial x}, f \mapsto \frac{\partial f}{\partial y} \Rightarrow \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are elts. of tangent space → comprise basis for tgt. space

dx and dy are defined so that $dx: \frac{\partial}{\partial x} \mapsto 1$, $dy: \frac{\partial}{\partial y} \mapsto 1$ "cotangent vector" $A\,dx + B\,dy$

Integration: you're adding up over little pieces of the surface/manifold.
Each little piece is determined by tgt. vectors spanning the element
form takes this set of tgt. vectors to a # that provides the "answer"

TOPS 35500