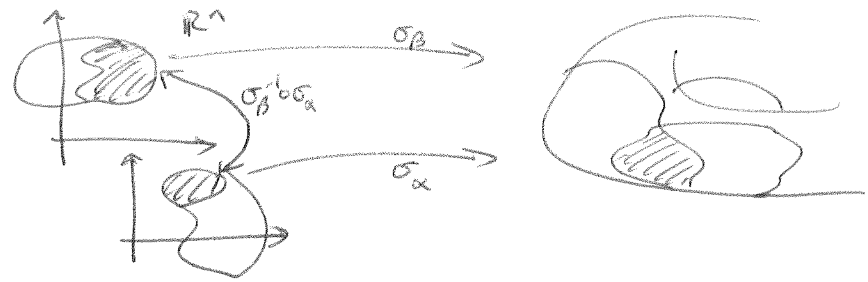
Tangent Spaces on Manifolds

Review

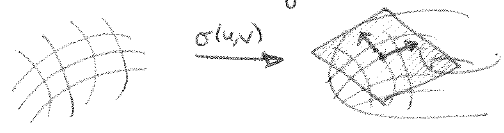
Recall: An manifold is a set $\mathcal{M}$ covered by open sets $\vec{\sigma}_\alpha : \mathcal{U}_\alpha \to \mathcal{M}$ ($\mathcal{U}_\alpha \subset \mathbb{R}^n$) such that the coordinate transformations $\vec{\sigma}_\beta^{-1} \circ \vec{\sigma}_\alpha$, where defined, are differentiable.



The Problem

Q: How do you define a tangent space/ tangent vector??

Normal:

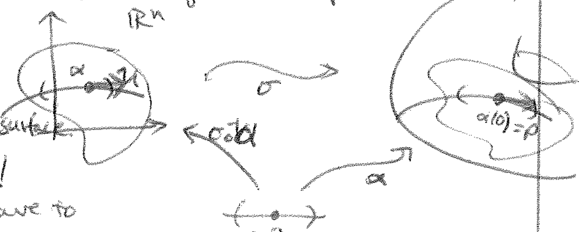


generated by $\vec{\sigma}_u$ and $\vec{\sigma}_v$, which can be defined when the manifold is in a Euclidean space.

Part 1:

A: Define the tangent vectors as existing in the pre-image of σ.

- Suppose $\alpha : (-\epsilon, \epsilon) \to \mathcal{M}$ is a differentiable curve in $\mathcal{M}$ through $\alpha(0) = p$.
- Then for a patch $\sigma : \mathcal{U} \to \mathcal{M}$, $\sigma^{-1} \circ \alpha(t) = (x_1(t), \dots, x_n(t))$ represents a curve in $\mathbb{R}^n$.
- Can identify the tangent vector with $(x_1'(0), \dots, x_n'(0))$
- ~~Unfortunately, this doesn't tell us anything about the surface.~~



Dir'l Derivatives

Part 2: This can't really be interpreted as a velocity!! So what do we do with the actual curve?? we have to rethink our understanding of vectors.

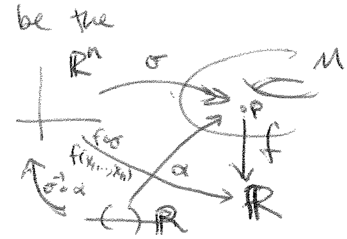
① Note that $\vec{v} \in \mathbb{R}^n \iff D_{\vec{v}} f \Big|_{t=0} = \vec{\nabla} f \cdot \vec{v} \Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Big|_{t=0} v_i = \left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \right) f.$

Use functions instead of vectors!!

Therefore, a tangent vector also defines a map $\mathcal{D} \to \mathbb{R}$ where $\mathcal{D}$ is the space of differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$. Every such map $\mathcal{D} \to \mathbb{R}$ can be written in terms of the $\frac{\partial}{\partial x_i}$, which comprise its basis. These are the important pieces... f is not really that vital.

② Since $(x_1'(0), \dots, x_n'(0))$ represents a vector in the premage, we may take $\mathcal{D}$ to be the space of fns $\mathcal{M} \to \mathbb{R}$ that are differentiable at p, and...

define the tangent vector to the curve α at $t=0$ to be the operator $\alpha'(0) = \sum_i^1 x_i'(0) \left(\frac{\partial}{\partial x_i} \right)_0$.



This takes an $f \in \mathcal{D}$ to $\sum_i x_i'(0) \left(\frac{\partial f}{\partial x_i} \right)_0$.

Otherwise put, $f \circ \alpha : \mathbb{R} \to \mathbb{R}$, so we may write $\alpha'(0) f = \frac{d(f \circ \alpha)}{dt} \Big|_{t=0}$, and view $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ as a function on functions.

Writing f in terms of local coords as $f(x_1, \dots, x_n)$, we get $\alpha'(0) f = \frac{d}{dt} f(x_1(t), \dots, x_n(t)) \Big|_{t=0}$

$$= \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial f}{\partial x_i} \right)_0 .$$

chain rule.

So we switch from $f \circ \alpha$ to $(f \circ \sigma) \circ (\sigma^{-1} \circ \alpha)$, so passing through $\mathbb{R}^n$ instead of $\mathcal{M}$.

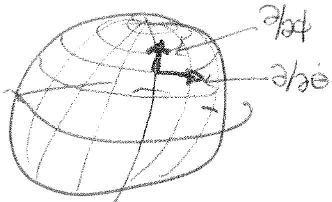
$\therefore$ we filter the derivative thru $\mathbb{R}^n$ instead of M, but use that to define a map on functions $f: M \to \mathbb{R}$, which provides a suitable representation of the ~~derivative~~ tangent vector.

We have a basis $\{ (\frac{\partial}{\partial x_1})_0, \ldots, (\frac{\partial}{\partial x_n})_0 \}$ of fns in $\mathcal{D} \to \mathbb{R}$ that defines the tangent space T_pM. It is a vector space!

Forms

Forms: The $(\frac{\partial}{\partial x_i})_0$ is the tangent vector associated w/ a coordinate curve.

e.g.



They're not really defined as these vectors, but are in every way equivalent.

$T_pM \cong \mathbb{R}^n$ .... they are the same space.

Differential Forms are diff'ble, linear functions from $T_pM \to \mathbb{R}$, i.e. they map $\sum_i v_i \frac{\partial}{\partial x^i} \longrightarrow \mathbb{R}$

These comprise the cotangent space, which is spanned by the dual basis $\{dx^1(0), \ldots, dx^n(0)\}$.

Eg. $ds^2 = E du^2 + 2F du dv + G dv^2$ is a 2-form, taking pairs of tangent vectors to specific values, where $E, 2F, G$ determine the length in each direction of the vector.

Thus, $\frac{\partial}{\partial u}, \frac{\partial}{\partial u} \to E$, $\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \to 2F$, etc.
It is the "length form" since $a, b \to a \cdot b$ the inner product; $a, a \to \|a\|^2$.

Forms useful eg. "Stoke's Theorem" $\int_P d\omega = \int_{\partial P} \omega$
"exterior derivative" bdry a diff form

Vector Fields

has own differentiable structure

A vector field is an assignment $p \mapsto v \in T_pM \;\forall\; p \in M$.

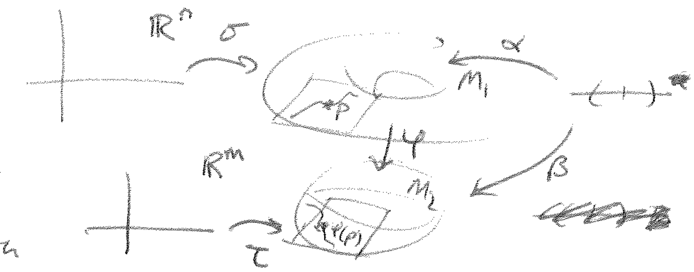
Or: it is a mapping $M \to TM$ where $TM = \{(p,v) : p \in M, v \in T_pM\}$ is the tangent bundle.

Vector fields map $X: \mathcal{D} \to \mathcal{F}$ mapping space of diff'ble fns on M to set of fns. on M.

$f \mapsto Xf$ where $(Xf)(p) = \sum_i a_i(p) \frac{\partial f}{\partial x_i}(p)$ ... extends idea of "directional derivative"

These are used to define all sorts of interesting things!

Maps between Manifolds



Let $\varphi: M_1 \to M_2$.
Define $d\varphi_p: T_pM_1 \to T_{\varphi(p)}M_2$ as follows:

① choose diff. curve $\alpha: (-\epsilon,\epsilon) \to M$ with $\alpha(0) = p$, $\alpha'(0) = v$.
② let $\beta = \varphi \circ \alpha$, set $d\varphi_p(v) = \beta'(0)$.

Defn: $d\varphi_p$ is the differential of φ at p, and is independent of local coords.

Defn: $\varphi: M_1 \to M_2$ is a diffeomorphism if it is differentiable, bijective, and φ^{-1} differentiable,
$\iff d\varphi_p$ is always an isomorphism.

Next: Riemannian Metrics on Manifolds (provide notion of distance).