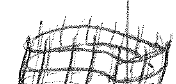
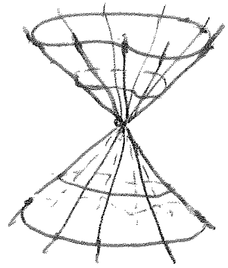
4.4 Surface Examples



① Generalized cylinder: $\vec{\sigma}(u,v) = \vec{\gamma}(u) + v\vec{a}$ for a curve $\vec{\gamma}$ and a vector $\vec{a}$.

- Regular $\Leftrightarrow$ tangent vector of $\vec{\gamma}$ is never parallel to $\vec{a}$ (implies also no line parallel to $\vec{a}$ intersects $\vec{\gamma}$ more than once)
- Can reduce parametrization to the general form $\vec{\sigma}(u,v) = (f(u), g(u), v)$.
- Coordinate patches require "patches" for $\vec{\gamma}(u)$ ... at least 2 if it's a closed curve.

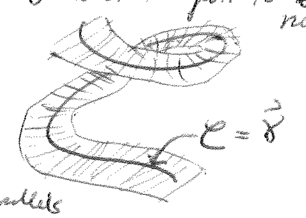
② Generalized cone: $\vec{\sigma}(u,v) = (1-v)\vec{p} + v\vec{\gamma}(u)$ for a curve $\vec{\gamma}$ and a point $\vec{p}$.



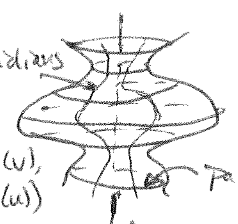
- $\vec{\gamma}$ should not pass through $\vec{p}$ $\Leftrightarrow$ regular
  no line through $\vec{p}$ intersects $\vec{\gamma}$ twice
  no tangent line to $\vec{\gamma}$ passes through $\vec{p}$
  and vertex is omitted.
- Parametrization reduces to $\vec{\sigma}(u,v) = (f(u)v, g(u)v, v)$
  w/ vertex at origin, curve $(f(u), g(u))$ in plane $z=1$.

③ Ruled surface: $\vec{\sigma}(u,v) = \vec{\gamma}(u) + v\vec{\delta}(u)$ where $\vec{\gamma}$ is a curve passing thru each line of the surface, and $\vec{\delta}(u)$ is in the direction of the lines

- regular $\Leftrightarrow$ $\dot{\vec{\gamma}} + v\dot{\vec{\delta}} \not\parallel \vec{\delta}$, i.e. if $\dot{\vec{\gamma}} \not\parallel \vec{\delta}$ and v is small. $\Leftrightarrow$ $\vec{\gamma}$ never tangent to $\vec{\delta}$ rulings



④ Surface of revolution: obtained by revolving a curve



- can reduce parametrization to $\vec{\sigma}(u,v) = (f(u)\cos v, f(u)\sin(v), g(u))$
- regular $\Leftrightarrow$ ① $f(u) \neq 0$
  ② $\vec{\gamma}$ is regular
- also must consider cases of self-intersection.

4.5 Quadric Surfaces (implicit surface)

Defined by $r^T A r + b^T r + c = 0$ $\Leftrightarrow$ $(x\ y\ z)\begin{pmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + (b_1\ b_2\ b_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + c = 0$

$\Leftrightarrow a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5yz + 2a_6xz + b_1x + b_2y + b_3z + c = 0.$

Classification:
(i) ellipsoid
(ii) hyperboloid 1-sheet
(iii) hyperboloid 2-sheets
(iv) elliptic paraboloid
(v) hyperbolic paraboloid
(vi) quadric cone
(vii) elliptic cylinder
(viii) hyperbolic cylinder
(ix) parabolic cylinder
(x) plane
(xi) two parallel planes
(xii) two intersecting planes
(xiii) straight line
(xiv) single point.

Pf: Relies on diagonalizability of real symmetric matrices.

Quadratic Forms

Thm: A symmetric $\Rightarrow$ different eigenvectors (from different eigenvals) are orthogonal.

Pf: $\lambda_1 \vec{v}_1 \cdot \vec{v}_2 = A\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2 \vec{v}_1 \cdot \vec{v}_2 \iff \vec{v}_1 \cdot \vec{v}_2 = 0.$ ∎

Thm: A is orthogonally diag'ble $\iff$ it is symmetric.

Pf: Ortho diag. $\Rightarrow A = PDP^T = PDP^{-1} \Rightarrow A^T = PD^T P^T = PDP^T = A$ since D diagonal.
By the above symmetric $\Rightarrow \exists$ orthogonal evects obtained via similarity. ∎

$\Rightarrow$ "**Spectral Decomposition**" $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$ for ~~orthogonal~~ orthonormal vectors $u_1, \ldots, u_n$.

Def: Quadratic Form is $Q(\vec{x}) = \vec{x}^T A \vec{x} = (x_1\ x_2 \cdots x_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
for A symmetric

- Then $x^T A x = y^T P^T A P y = y^T D y$, i.e. $Q(x) = a_{11} y_1^2 + a_{22} y_2^2 + \cdots + a_{nn} y_n^2$
  for appropriate change of basis $\vec{y} = P^T \vec{x}$.

Reduction of Quad Sfcs: $r^T A r + b_1^T r + c_1 = 0 \rightsquigarrow r^T D r + b_2^T r + c_2 = 0$

$\rightsquigarrow (a_1 y_1^2 + b_1 y_1) + (a_2 y_2^2 + b_2 y_2) + \cdots + c = 0$

can't complete square $\leftsquigarrow$ $\rightsquigarrow$ completing the square

$a_1 (y_1 - \frac{b_1}{2a_1})^2 + \cdots + c = 0$

$\rightsquigarrow$ shifting $a_1 y_1^2 + \cdots + a_n y_n^2 = c$.

$\rightsquigarrow$ dividing by c $\begin{cases} a_1 y_1^2 + \cdots + a_n y_n^2 = 1 \\ a_1 y_1^2 + \cdots + a_n y_n^2 = 0. \end{cases}$

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