# Compactness w/ Appl. to Fixed Point Theorems 4/15/09 ①

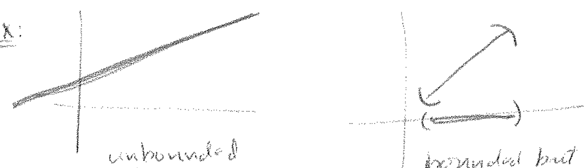
**Compactness** $\Rightarrow$ on $\mathbb{R}^n$, compact $\Leftrightarrow$ closed + bounded    cpt is the "next best to finite"

Def 1: Compact if every open cover has a finite subcover (covering compact)
Def 2: Compact if every infinite subset of $X$ has a limit point (limit point compact)

Facts: Def 1 $\Rightarrow$ Def 2, but not necessarily the opposite
In a metric space, where the topology is defined by a distance fn, they are equivalent

Thm: (EVT) If $f: X \to \mathbb{R}$ is cts, w/ $X$ compact, then $f$ has max/min values, ie. $\exists\, a, b$ s.t. $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.

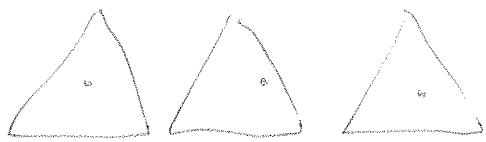
Ex:



unbounded domain no max/min.

bounded but not closed domain no max/min.

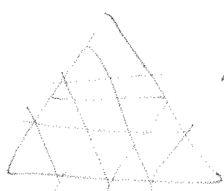
Ex:



Any sequence of points in the triangle converges.

**Brouwer Fixed Pt Thm:** Every cts fn $f: D \to D$ from disk to itself has a fixed pt.

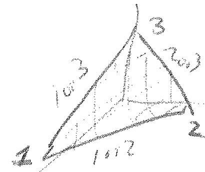
Pf:



At each pt $(a,b,c)$, mapped to $f(a,b,c) = (a',b',c')$ ← "Barycentric coords" that sum to 1.

label 1 if $a' < a$
2 if $a' \geq a$ but $b' < b$
3 if $a' \geq a$, $b' \geq b$, but $c' < c$.

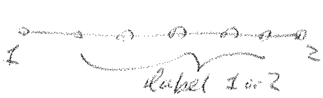
Else $a' \geq a$, $b' \geq b$, $c' \geq c \Rightarrow$ equality.




The labeling forces a Sperner labeling, implying a convergent set of triangles, and therefore a limit point (by compactness).

So: continuity forces $x' \leq x$ at ① label, $y' \leq y$ at ② label, $z' \leq z$ at ③ label, so that $f(x,y,z) = (x,y,z)$ fixed pt. ∎

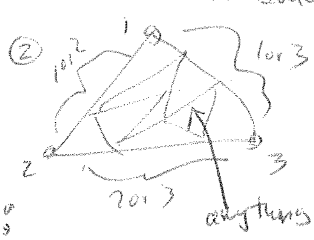
**Sperner's Lemma:** ① Base case:



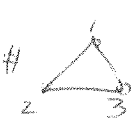

$\Rightarrow$ odd # $\overset{}{1}\!-\!\overset{}{2}$ edges

"Discrete Intermediate Value Theorem"

②




$\Rightarrow$ odd #




triangles (inductive proof).

⋮

generalizes!!