4/13/09

Riemannian Geometry

Idea: remove dependence of sfc. defn. on $\mathbb{R}^3$

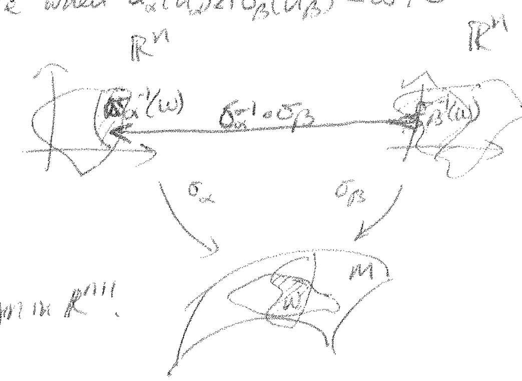
Recall: reg. sfc: has coordinate patches $\mathcal{U} \subset \mathbb{R}^2$ (open) $\to \mathbb{R}^3$ (set in $\mathbb{R}^3$), st. tangent plane exists at every point
overlapping patches are "compatible"
$\hookrightarrow$ properties of $\mathbb{R}^3$ give rise to normal vector, area, length, etc.

1854 Riemann
1913 H. Weyl

New Approach:

Defn: Diff. mfd. (dim. n) is a set M (not inside another space) and a family $\vec{\sigma}_\alpha : \mathcal{U}_\alpha \subset \mathbb{R}^n \to M$ (open; coord. patches)
such that
① $\bigcup_\alpha \vec{\sigma}_\alpha(\mathcal{U}_\alpha) = M$ covers the whole set
② the transition maps $\vec{\sigma}_\beta^{-1} \circ \vec{\sigma}_\alpha$ are differentiable when $\vec{\sigma}_\alpha(\mathcal{U}_\alpha) \cap \vec{\sigma}_\beta(\mathcal{U}_\beta) = W \neq 0$
③ the family is "maximal" ie. contains all possible patches.

The family is an atlas or differentiable structure.



Note: Here we express coord. transf. as an axiom, rather than a property.

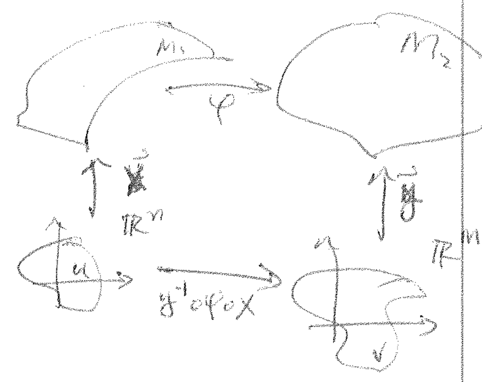
Ex: Real projective space $\mathbb{RP}^n$ = set of lines thru origin in $\mathbb{R}^{n+1}$.

$\hookrightarrow \mathbb{RP}^n = \{ \vec{x} \in \mathbb{R}^{n+1} : \vec{x} \sim \lambda \vec{x} \ \forall \lambda \neq 0, \lambda \in \mathbb{R} \}$

$V_i = \{ [x_1, \dots, x_{n+1}] \in \mathbb{RP}^n ; x_i \neq 0 \}$

$\vec{X}_i : (y_1, \dots, y_n) \mapsto [y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n]$ comprises family of patches,
can show differentiability:
$X_j^{-1} \circ X_i (y_1, \dots, y_n) = \left(\frac{y_1}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \dots, \frac{y_n}{y_j} \right)$ diff'ble since $y_j \neq 0$.

Defn: $\varphi : M_1^n \to M_2^m$ differentiable at $p \in M_1$
if $y^{-1} \circ \varphi \circ x$ is always differentiable.



We will replace the $\mathcal{F}_I$ which gives length with an inner product on the tangent space.

$\int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt \longrightarrow \int_a^b \sqrt{\left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle}\, dt$ for a certain defn. of derivative.

But how to define tangent vectors?