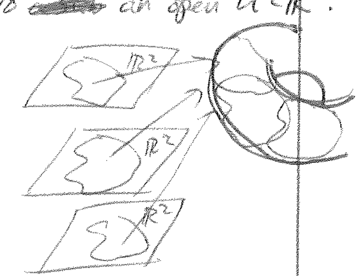
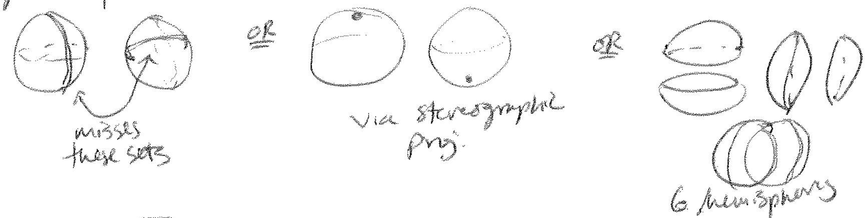
4.1-4.2

Surface: set $S \subset \mathbb{R}^3$
for every pt. $p \in S$ $\exists$ open nbhd $S \cap W$ of p homeomorphic to an open $U \subset \mathbb{R}^2$.
"atlas of patches".. generally assume patches are connected.



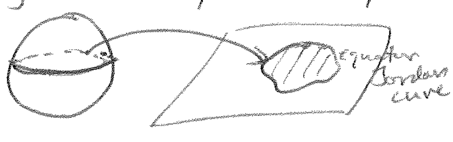
Ex: Sphere patches



Ex: Cone

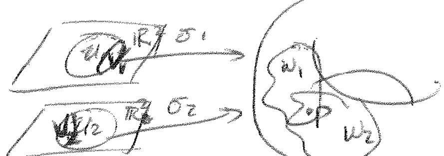
Not a surface b/c contains a 1D point.
E.g. in $\mathbb{R}^2$, remove a pt. of an open set doesn't change path connectedness but a nbhd of the cone pt. it does!

Ex: Why can't do 1 patch on a sphere?



?? counterex. to Jordan Curve Thm.

Def: If $p \in S$ is in 2 patches, have



Let $V_1 = \sigma_1^{-1}(W_1 \cap W_2)$, $V_2 = \sigma_2^{-1}(W_1 \cap W_2)$

Then, $\sigma_2^{-1} \circ \sigma_1 : V_1 \to V_2$ is a homeomorphism and called the coordinate transition map.

Def: Sfc Patch $\vec{\sigma}: U \to \mathbb{R}^3$ is regular if smooth and $\vec{\sigma}_u, \vec{\sigma}_v$ are linearly independent (i.e. non-zero and non-parallel or $\vec{\sigma}_u \times \vec{\sigma}_v$ always nonzero.)

$\boxed{\textbf{Def: } \text{Smooth Surface} = \text{atlas of smooth, regular patches}}$ * Can assume all patches are included in the atlas.

Ex: Plane $\vec{\sigma}(u,v) = \vec{a} + u\vec{p} + v\vec{q}$, so long as $\vec{p}, \vec{q}$ are nonzero, nonparallel.

Props: (1) Transition Map of Smooth Sfc. are Smooth (consequence of inv. fn. thm.)

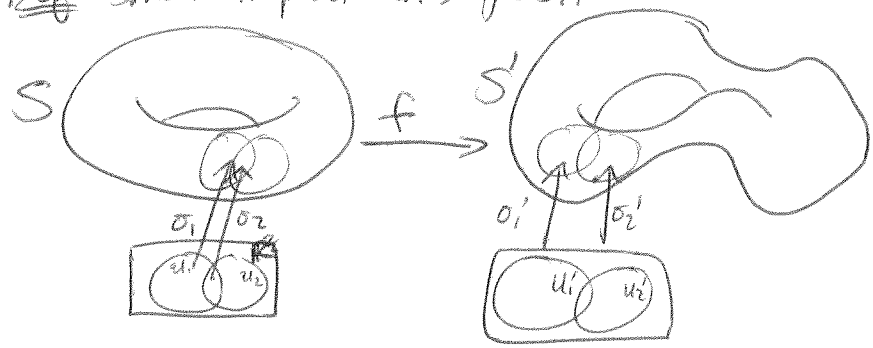
(2) Conversely if $S \xleftarrow{\sigma_1}$  $\sigma_1 : U_1 \to \mathbb{R}^3$ a regular patch. U_1, U_2 are diffeomorphic via Φ, i.e. Φ is smooth homeomorphism w/ smooth inverse,

then $\sigma_2 = \sigma_1 \circ \Phi : U_2 \to S$ is a smooth, regular patch.

Pf: ⟨Bcs. regularity, smoothness both kept under inversion + compositions⟩

Def: Such patches are reparametrizations.

Def: Smooth map between surfaces??



Smooth iff all patches w/ composition $\sigma_j'^{-1} \circ f \circ \sigma_i$ are smooth.

Well-defined since $\sigma_j'^{-1} \circ f \circ \sigma_i$ smooth $\Leftrightarrow$ reparam. $\tilde{\sigma}_j'^{-1} \circ f \circ \tilde{\sigma}_i$ smooth.

Defn: If $f: S_1 \to S_2$ is bijective, both f, f^{-1} smooth, then its a diffeomorphism + surfaces are diffeomorphic.

* This is the proper notion of equivalence among surfaces in diff. geo.
Lots of geometric props. respect this notion.

Thm: Spse. $\exists$ subset $S \subset \mathbb{R}^3$ s.t. each point has a nbhd that is part of a smooth level fn. w/ partial derivatives not all zero, $f: \mathbb{R}^3 \to \mathbb{R}$, then its a smooth surface.

(i.e. level surfaces are smooth surfaces)

Next Time: ~~§§3~~-4.5